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OPTIMAL PROPORTIONAL REINSURANCE WITH A CONSTANT RATE OF INTEREST

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Introduction:

The classical measure for an insurance risk is the ruin probability. This is the probability that the surplus process of an insurance company becomes negative in finite time. Ruin probabilities are, from the perspective of a risk manager, the natural dynamic counterpart of the value at risk. We say that ruin occurs when the surplus process, modelled as a stochastic process, becomes negative for the first time. The ruin probability indicates the soundness of the insurer's combination of the income of an insurance company plus the initial capital on the one hand and the claims process on the other. Also, we obtain a useful tool for portfolio comparison. But despite these positive points, the use of ruin probabilities has been criticised. For instance, the ruin probability does not take into account the time of ruin nor the severity of ruin. Let X be the underlying surplus process with $X_0 = x$. Let Y be an increasing process with $Y_{0-} = 0$. The process with capital injections is denoted by $X_t^Y = X_t + Y_t$. We define the value $V^Y(x) = E_x[\int_0^\infty e^{-\delta t} dY_t]$, where $\delta \geq 0$. The injection process Y has to be chosen such that $X_t^Y \geq 0$ for all t (almost surely). The value function is defined as V(x)= inf $V^Y(x)$, where the infimum is taken over all cadlag processes Y such that $X_t^Y \geq 0$ for all t. Because of the discounting or because ruin is not certain, it is not optimal to inject capital before it is really necessary. We allow the insurer to invest the positive excess in a riskless asset with a constant interest rate *m*. We are also interested in finding the optimal reinsurance strategy and the value function as the infimum of all possible expected discounted capital injections due to admissible reinsurance strategies.

Ruin:

The event that the process $X_t = x + ct - \sum_{i=1}^{N_t} Z_i$ is defined, ever falls below zero is called ruin. The time T when the process X_t falls below zero for the first time. T = inf {t > 0: X_t > 0} is called ruin time. As severity of ruin one denotes the absolute value $|X_T|$. The probability of ruin for the initial capital x is then given by $\chi(x) = P[T < \infty / X_0 = x]$

Return and Value Functions:

The risk measure connected to some admissible strategy pair (A,B). We choose the value of expected discounted capital injections with some discounting factor $\delta \geq 0$.

$$V(x) = \inf_{(A,B)} V^{A,B}(x) = \inf_{(A,B)} E_x[\int_0^\infty e^{-\delta t} dY_t^{A,B}]$$

Brownian Motion Process:

The stochastic process {x(t), t≥0} is called a Brownian motion process. if, (i) X(t) has independent increments, i.e., for every pair of disjoint intervals of time (s,t) and (u,v), where s≤t≤u≤v, the random variables {x(t)-x(s)} and {x(v)-x(u)} are independent. (ii) Every increment {x(t)-x(s)} is normally distributed with mean μ(t-s) and variance σ^2(t-s).

Standard Brownian Motion:

A Brownian motion process {x(t), t≥0} with x(0)=0, μ=0, σ=1 is called a standard Brownian motion process.

Bounded Variation:

A real-valued function f on the real line is said to be Bounded Variation (BV function) on a chosen interval [a,b] ⊂R if its total variation is finite. i.e., $f \in Bv([a.b]) \Leftrightarrow V_a^b(f) < +\infty$

Proportional Reinsurance for a Diffusion Approximation:

Consider the surplus process of an insurance company, where the time horizon is infinite:

$$C_t = x + ct - \sum_{i=1}^{N_t} Z_i \qquad (1)$$

Here $\{N_t\}$ is the Poisson process with intensity $\lambda > 0$ and $\{Z_i\}_{i \in N}$ is a sequence of independent and identically distributed random variables. The Z_i are assumed to have a distribution G with $\mu = E[Z_i]$ and $\mu_2 = E[Z_i^2] < \infty$, and to be independent of $\{N_t\}$. The premium income of the insurer is $c=(1+\eta)\lambda\mu$ for some $\eta>0$. Furthermore, the insurer can buy proportional reinsurance. That is, the insurer has to choose a retention level $b\in[0,1]$ and the reinsurer carries (1-b) Z_i from each claim Z_i. The premium rate remaining to the insurer calculated by an expected value principle is $c(b)=\lambda\mu b(1+\theta)-\lambda\mu(\theta-\eta)$, where θ is the safety loading of the reinsurer. In order to avoid the case where the insurer can get rid of the risk by buying full reinsurance and still receiving a non negative premium, We assume that $\theta>\eta$. The insurer can change his retention level continuously. A diffusion approximation to the above classical risk model then fulfils the stochastic differential equation. $dX_t^B=\{\lambda\mu[b_t\theta-(\theta-\eta)]\}dt+b_t\sqrt{\lambda\mu_2}dW_t$, Where $\{W_t\}$ is a standard Brownian motion. In the section we work on a probability space (Ω, F, P) containing the Brownian motion $\{W_t\}$. We call the reinsurance strategy $B=\{b_t\}$ admissible if it is adapted and cadlag, and $b_t\in[0,1]$ for all t; the set of all reinsurance strategies is denoted by U. Since b_t is bounded and cadlag, the integrals are well defined. Now we allow the insurer to earn interest on positive surplus with a constant force of interest. It is clear that if X is at 0, we must inject capital to stop the process entering $(-\infty,0)$. We interpret Y_t as the cumulative capital injections up to time t and associate with $Y^B=\{Y_t^B\}$ the controlled process with capital injections.

$$dX_t^{B,Y,m}=\{mX_t^{B,Y,m}+\lambda\mu[b_t\theta-(\theta-\eta)]\}dt+b_t\sqrt{\lambda\mu_2}dW_t+dY_t^B$$

for a constant interest rate m>0. Because the preference rate δ is nonnegative, we should inject capital only when the process becomes negative and only enough to allow the process to shift to 0 again [3]. That is, the smallest non decreasing process $\{Y_t\}$ such that $X_t^{B,Y,m}\geq 0$ for all t. It is well known that the process Y exists. Note that as a non decreasing process, $Y^B=\{Y_t^B\}$ is of bounded variation. We want to measure the risk, connected to some reinsurance strategy B, by the expected discounted capital injections $V^B(x)=E_x[\int_0^\infty e^{-\delta t}dY_t^B]$. Our goal is to find the value function by minimising $V^B(x)$ over all admissible reinsurance strategies:

$$V(x):=\inf V^B(x)$$

It would seem natural that $\delta\geq m$. Indeed, if $\delta<m$, the capital injections would be discounted at a lower rate than the surplus. However, we do not make a restriction, and allow all $\delta\geq 0$ and $m>0$. It is clear that the value function V(x) is decreasing. In particular, we obtain, for the constant strategy $B\equiv 0$ before ruin occurs,

$$X_t^{0,m}=x-\lambda\mu(\theta-\eta)t+m\int_0^t X_s^{0,m}ds$$
$$=(x-\lambda\mu(\theta-\eta)m^{-1})e^{mt}+\lambda\mu(\theta-\eta)m^{-1}$$

Since it holds that $X_t^{0,m}>0$ for all t if $x\geq\lambda\mu(\theta-\eta)$, we conclude that $\{Y_t^0\}\equiv 0$. Therefore, V(x)=0 for $x\geq\lambda\mu(\theta-\eta)$. Thus, we have to consider only $0\leq x<\lambda\mu(\theta-\eta)m^{-1}$.

Remark:

Let $\{X_t\}$ be a process fulfilling the stochastic differential equation $dX_t=a(X_t)dt+\sigma(X_t)dW_t$, Where a and σ are functions such that the above equation has a unique strong solution. The process with capital injections then fulfils $dX_t^Y=a(X_t^Y)dt+\sigma(X_t^Y)dW_t+dY_t$, Whereas Y is the local time of the process at 0. The corresponding return function $V(x)=E_x[\int_0^\infty e^{-\delta t}dY_t]$ solve the differential equation

$$\tfrac{\sigma^2(x)}{2}V''(x)+a(x)V'(x)-\delta V(x)=0 \quad for \quad x\geq 0,$$

and fulfils V′(0)=-1 and $\lim_{x\to\infty}V(x)=0$ [4]. we also know that every solution f(x) to the above differential equation, vanishing at ∞, has the form $f(x)=f'(0)E_x[\int_0^\infty e^{-\delta t}dY_t]$. Now equipped with the knowledge of how to calculate the return function for a given reinsurance strategy B, we illustrate the method by an example.

Example:

Consider a constant strategy $B\equiv b\in[0,1]$ the corresponding return function $V^b(x)$ solves the differential equation $\frac{b^2\lambda\mu_2}{2}f''(x)+(mx+\lambda\mu(b\theta-(\theta-\eta)))f'(x)-\delta f(x)=0$ With the power series method we find that solutions to the above differential equation are given by

$$C_1\left(1+\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}\left(\frac{\delta}{m+2-2k}\right)}{(2n)!}\left(\frac{2m}{\lambda\mu_2 b^2}\right)^n\left(x+\lambda\mu(b\theta-\theta+\eta)m^{-1}\right)^{2n}\right)$$

$$+C_2\left(\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}\left(\frac{\delta}{m+1-2k}\right)}{(2n+1)!}\left(\frac{2m}{\lambda\mu_2 b^2}\right)^n (x+\lambda\mu(b\theta-\theta+\eta)m^{-1})^{2n+1}+x+\lambda\mu(b\theta-\theta+\eta)m^{-1}\right)$$

Using the initial conditions $\lim_{x\to\infty}V^b(x)=0$ and $(V^b)'(0)=-1$, we can calculate the coefficients C_1 and C_2. Let, for example, $b=0.5, \lambda=\mu=1, \mu_2=2, \theta=0.5, \eta=0.3, \delta=0.04,$ and m=0.03. Then we obtain

$$C_1=4.084921164 \text{ and } C_2=-1.947322694$$

Changing the parameter θ to $\theta=0.8$ yields $C_1=0.9686572638$ and $C_2=-0.4617685869$ Recall that V(x)=0 for $x\geq\lambda\mu(\theta-\eta)m^{-1}$. For $x<\lambda\mu(\theta-\eta)m^{-1}$, the Hamilton - Jacobi-Bellman(HJB) equation is given by

$$\frac{1}{2}\lambda\mu_2 b^2 V''(x)+\{mx-\lambda\mu(\theta-\eta)+\lambda\mu b\theta\}V'(x)-\delta V(x)=0$$

We abandon the explicit derivation of the HJB equation. Note that if the value function is twice continuously differentiable and solves the HJB equation above, it must be strictly convex. In fact, choosing $\hat{b}=1-\frac{\eta}{\theta}-\frac{mx}{\lambda\mu\theta}$ (note that $\hat{b}\in[0,1]$) we obtain $\frac{1}{2}\lambda\mu_2\hat{b}^2 V''(x)-\delta V(x)\geq 0$. Since V(x)>0, convexity follows,

We make the ansatz $V(x)=C(m^{-1}\lambda\mu(\theta-\eta)-x)^k$ For some C > 0 and k > 1. Then (1.2) reads

$$0=\frac{1}{2}\lambda\mu_2 b^2 k(k-1)(m^{-1}\lambda\mu(\theta-\eta)-x)^{k-2}-\{mx+\lambda\mu[b\theta-(\theta-\eta)]\}$$
$$k(m^{-1}\lambda\mu(\theta-\eta)-x)^{k-1}-\delta(m^{-1}\lambda\mu(\theta-\eta)-x)^k$$

The optimal b is then given by $b(x)=\frac{\theta\mu(m^{-1}\lambda\mu(\theta-\eta)-x)}{\mu_2(k-1)}$, (3)

Provided that $b(x)\leq 1$, i.e. x is close enough to $m^{-1}\lambda\mu(\theta-\eta)$. If b(x)>1, no reinsurance has to be chosen. Plugging in the optimal b(x) and dividing by $(m^{-1}\lambda\mu(\theta-\eta)-x)^k$, we find that

$$mk-\frac{\lambda k\theta^2\mu^2}{2\mu_2(k-1)}-\delta=0 \quad (4)$$

Solving for *k* yields the solution

$$k=\frac{\delta\mu_2+m\mu_2+\frac{\lambda\theta^2\mu^2}{2}+\sqrt{\left(\delta\mu_2+m\mu_2+\frac{\lambda\theta^2\mu^2}{2}\right)^2-4m\mu_2^2\delta}}{2m\mu_2} \quad (5)$$

Note that the other solution is smaller than 1.

Remark:

Let $X^*=X^{b(x),Y,m}$ with initial value $0\leq x<\lambda\mu(\theta-\eta)m^{-1}$, where the reinsurance strategy b(x) is given in (1.3). Consider the process $Z_t=m^{-1}\lambda\mu(\theta-\eta)-X_t^*$ for $t\in[0,\tau^*]$, where $\tau^*=\inf\{s:X_s^*=0\}$.

Then $$dZ_t=-\frac{\sqrt{\lambda\mu_2}\theta\mu}{\mu_2(k-1)}z_t dW_t-\left(\frac{\lambda\mu^2\theta^2}{\mu_2(k-1)}-m\right)z_t dt$$

This means that $\{Z_t\}$ is a geometric Brownian motion. Taking the logarithm gives

$$d(\log(Z_t))=-\frac{\sqrt{\lambda\mu_2}\theta\mu}{\mu_2(k-1)}dW_t+\left(m-\frac{\lambda\mu^2\theta^2(2k-1)}{2\mu_2(k-1)^2}\right)dt,$$

In particular, the surplus X^* will never reach the value $m^{-1}\lambda\mu(\theta-\eta)$, where full reinsurance would be bought. The considerations we used in deriving (1.3) are of a heuristic nature.

Theorem:

Define $\tilde{x}:=\{m^{-1}\lambda\mu(\theta-\eta)-\mu_2(k-1)/\theta\mu\}\vee 0$ Then the strategy

$$b^*(x)=\begin{cases}0, & x\geq m^{-1}\lambda\mu(\theta-\eta),\\ b(x), & \tilde{x}<x<m^{-1}\lambda\mu(\theta-\eta),\\ 1, & x\leq\tilde{x}\end{cases},$$

Where b(x) is given in (3), is an optimal reinsurance strategy. The function f(x), given by

$$f(x)=\begin{cases}0, & x\geq m^{-1}\lambda\mu(\theta-\eta),\\ f_2(x), & \tilde{x}\leq x<m^{-1}\lambda\mu(\theta-\eta),\\ f_1(x), & 0<x<\tilde{x},\\ f(0)-x, & x\leq 0,\end{cases}$$

With $f_1(x) = C_1\left(1+\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}(\delta/m+2-2k)}{(2n)!}\left(\frac{2m}{\lambda\mu_2}\right)^n (x+\lambda\mu\eta m^{-1})^{2n}\right)+C_2\left(x+\lambda\mu\eta m^{-1}+\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}(\delta/m+1-2k)}{(2n+1)!}\left(\frac{2m}{\lambda\mu_2}\right)^n (x+\lambda\mu\eta m^{-1})^{2n+1}\right)$

and $f_2(x) = C_3(r^{-1}\lambda\mu(\theta-\eta)-x)^k$,

where k is given in (1.5) is twice continuously differentiable, solves the HJB equation (1.2), and f(x)=V(x). If $\tilde{x}>0$, the coefficients C_1, C_2 and C_3 are uniquely determined by the system of equations

$$f_1'(0)=-1,\ f_1'(\tilde{x})=f_2'(\tilde{x}),\ f_1''(\tilde{x})=f_2''(\tilde{x});$$

If $\tilde{x}\le 0$, C_3 is given by $f_2'(0)=-1$.

Proof:

We have already seen that f(x) solves the HJB equation (1.2) in the interval $(\hat{x}, m^{-1}\lambda\mu(\theta-\eta))$. From Example 1.1 we know that $f_1(x)$ solves (1.2) provided that $b^*(x)=1$. It therefore remains to show that the infimum really is attained at $b^*(x)=1$ for $x\in[0,\tilde{x}]$. Note that since $f_1'(\tilde{x})=f_2'(\tilde{x})$ and $f_1''(\tilde{x})=f_2''(\tilde{x})$, we have $b^*(\tilde{x})=1$. Assume that $\tilde{x}>0$, otherwise there is nothing to show. Note that

$\lambda\mu m^{-1}(\theta-\eta)-\frac{\mu_2(k-1)}{\theta\mu}\ge 0$ holds if and only if

$$\eta\le\theta\frac{1-\frac{2\delta}{mk}}{2\left(1-\frac{\delta}{mk}\right)}$$

Consider the HJB equation (1.2) with the function $f_1(x)$. Since the minimum is attained at $b^*(x)=\frac{-\mu\theta f_1'(x)}{\mu_2 f_1''(x)}\wedge 1$, we need to show that $g(x):=\frac{-\mu\theta f_1'(x)}{\mu_2 f_1''(x)}\ge 1$ for all $x\in[0,\tilde{x}]$ .Assume for the moment that there exists some $x\in[0,\tilde{x}]$ with g(x)<1. Because $g(\tilde{x})=1$ and g is continuous, there exist some interval $[a,b]\subset[0,\tilde{x}]$ and $x^*\in[a,b]$ such that g′(x) > 0 on [a,b] and $g(x^*)<1$.

So we know that $g'(x^*)=-\frac{\theta\mu}{\mu_2}+\frac{\theta\mu}{\mu_2}\frac{f'(x^*)f'''(x^*)}{f''(x^*)^2}>0$. It readily follows that

$$1<\frac{f_1'(x^*)f_1'''(x^*)}{f_1''(x^*)^2}=\frac{-\theta\mu f_1'(x^*)}{\mu_2 f_1''(x^*)}\ \frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)}=g(x^*)\frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)}$$

Because $g(x^*)$ was assumed to be smaller than 1, $\frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)}$ has to be larger than 1. The function $f_1(x)$ is smooth. From Example 1 we know that $f_1(x)$ fulfils the differential equation

$$\frac{\lambda\mu_2}{2}f_1''(x)+(mx+\lambda\mu\eta)f_1'(x)-\delta f_1(x)=0,$$

From which we obtain the following representation: $\frac{\lambda\mu_2}{2}f_1'''(x^*)+\{mx^*+\lambda\mu\eta\}f_1''(x^*)-(\delta-m)f_1'(x^*)=0$

Rearranging the terms, dividing by $f_1''(x^*)$ and using (1.4) yields, for $\delta\ge m$,

$$\frac{-\mu_2 f'''(x^*)}{\theta\mu f_1''(x^*)}=-(\delta-m)\frac{2f_1'(x^*)}{\lambda\theta\mu f_1''(x^*)}+\frac{2}{\lambda\theta\mu}\{mx^*+\lambda\mu\eta\}<(\delta-m)\frac{2\mu^2}{\lambda\theta^2\mu^2}+\frac{2}{\lambda\theta\mu}\{m\tilde{x}+\lambda\mu\eta\}$$

$$=(\delta-m)\frac{2\mu_2}{\lambda\theta^2\mu^2}+\frac{2}{\lambda\theta\mu}\left\{\lambda\mu\theta-\frac{m\mu_2(k-1)}{\theta\mu}\right\}=2-\frac{k}{k-1}$$

$$\frac{k-2}{k-1}<1,$$

Where we used the definition of *k*. This is a contradiction. For $m>\delta$, we also obtain a contradiction. Because $f_1'(x)\le 0$ and $f_1''(x)\ge 0$, we have, from the definition of *k*,

$$\frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)}<2-\frac{mk}{mk-\delta}<1.$$

Now we will show that f(x)=V(x). Consider an arbitrary admissible reinsurance strategy $B=\{b_t\}$, and define $\hat{X}_t=X^{B,Y,m}$. Then $\hat{x}$ is given by the differential equation $d\hat{X}_t=\{m\hat{X}_t+\lambda\mu[b_t\theta-(\theta-\eta)]\}dt+b_t\sqrt{\lambda\mu_2}dW_t+dY_t^B$ Let $x_n=m^{-1}\lambda\mu(\theta-\eta)-n^{-1}$. We suppose that n is large enough that $x_n>\tilde{x}$. Furthermore, let

$\tau_n = \inf\{t : \hat{X}_t > x_n\}$ and $\tau_0 = \lim_{n\to\infty} \tau_n = \infty$. Since f(x) is twice continuously differentiable, $\hat{X}_t \geq 0$, and f' (0)=-1, using (1.2), we apply $it\hat{o}$'s formula to the function $e^{-\delta t} f(x)$ to obtain

$$e^{-\delta(\tau_n\wedge t)} f(\hat{X}_{\tau_n\wedge t}) = f(x) + \int_0^{\tau_n\wedge t} e^{-\delta s} f'(\hat{X}_s) dY_s^B + \int_0^{\tau_n\wedge t} e^{-\delta s}\{D_{s,B} f(\hat{X}_s) - \delta f(\hat{X}_s)\} ds$$
$$+ \int_0^{\tau_n\wedge t} e^{-\delta s} f'(\hat{X}_s) b_s \sqrt{\lambda\mu_2}\, dW_s$$
$$\geq f(x) + \int_0^{\tau_n\wedge t} e^{-\delta s} dY_s^B + \int_0^{\tau_n\wedge t} e^{-\delta s} f'(\hat{X}_s) b_s \sqrt{\lambda\mu_2}\, dW_s,$$

Where $D_{s,B} f(x) = \frac{\lambda\mu_2 b_s^2}{2} f''(x) + \{mx + \lambda\mu(b_s\theta - \theta + \eta)\} f'(x)$

is the infinitesimal generator of the process $X_t^{b_s,m}$. Because the derivative of the value function is bounded, we can conclude that the stochastic integral is a martingale with zero expectation. Thus, taking expectations of both sides of the above inequality we have $f(x) \leq E_x[e^{-\delta(\tau_n\wedge t)} f(\hat{X}_{\tau_n\wedge t})] + E_x[\int_0^{\tau_n\wedge t} e^{-\delta s} dY_s^B]$

Letting $n \to \infty$ we obtain, using the fact that $f(\hat{X}_{\tau 0}) = 0$, $f(x) \leq e^{-\delta t} E_x[f(\hat{X}_t); \tau_0 > t] + E_x[\int_0^{\tau 0\wedge t} e^{-\delta s} dY_s^B$

Since f(x) is bounded, we can let $t \to \infty$, yielding $f(x) \leq E_x\left[\int_0^{\tau_0} e^{-\delta s} dY_s^B\right] \leq E_x\left[\int_0^{\infty} e^{-\delta s} dY_S^B\right]$

This implies that $f(x) \leq V(x)$. Repeating the calculations above with the proposed optimal strategy, the inequalities become equalities. This proves that f(x)=V(x).

The Classical Risk Model:

We consider the classical risk model (1.1). The probability space (Ω,F,P) is assumed to contain the compound Poisson process $\sum_{i=1}^{N_t} Z_i$. By Z we denote a generic random variable with the same distribution as Z_i. The insurer can buy reinsurance. In contrast to the previous section, we now allow more general reinsurance treaties. In the examples we will again return to proportional reinsurance. Choosing the level $b \in [0, \tilde{b}]$, the insurer pays $r(Z_i, b)$ for a claim of size Z_i. Here b=0 means 'full reinsurance' and $b = \tilde{b} \in (0, \infty]$ means 'no reinsurance'. For example, for proportional reinsurance, r(Z,b)=bZ and $b \in [0,1]$. For excess of loss reinsurance, we obtain $r(Z,b) = \min\{Z, b\}$ and $b \in [0, \infty]$. For the reinsurance cover, the insurer pays a premium at rate c-c(b). That is, the premium rate left for the cedent is c(b). For simplicity, we assume that r(z,b) is continuous and increasing in both z and b, and that c(b) is continuous and increasing with c(0)<0 and $c(\tilde{b}) = c$. The assumption that c(0)<0 is needed in order that the problem below is not trivial. For example, if the reinsurer uses an expected value principle with safety loading θ, we obtain

$$c(b) = c - (1+\theta)\lambda E[Z - r(Z,b)] = (1+\theta)\lambda E[r(Z,b)] - (\theta - \eta)\lambda\mu.$$

The condition c(0)<0 is fulfilled if $\theta > \eta$. The insurer can choose the level b_t at any time point t. Because no information about the future can be used, the process $\{b_t\}$ is assumed to be cadlag and adapted. The surplus of the insurer including reinsurance then has the form $X_t^B = x + \int_0^t c(b_s) ds - \sum_{i=1}^{N_t} r(Z_i, b_{T_i -})$. As before, the insurer can invest his money, if his surplus is positive, with a fixed rate of interest m>0. To prevent the surplus process becoming negative, the insurer has to inject additional capital. We denote the accumulated capital injections until time t by $\{Y_t^B\}$. The surplus process therefore has the form $X_t^{B,m,Y} = x + \int_0^t (c(b_s) + mX_s^{B,m,Y}) ds - \sum_{i=1}^{N_t} r(Z_i, b_{T_i -}) + Y_t^B$

We are interested in the minimal value $V(x) = \inf\ E[\int_0^\infty e^{-\delta t} dY_t^B]$. Because of the interest, we do not need to assume the net profit condition $\eta > 0$. Note that the process has deterministic paths between the claim times. Let $\{T_k\}$ denote the claim times. Then we have, for $t < T_1$, $X_t^{B,m} = x + \int_0^t c(b_s) + m\int_0^t X_s^{B,m} ds$, So that we can write $X_t^{B,m} = (\int_0^t c(b_s) e^{-ms} ds + x) e^{mt}$.

Lemma:

The function V(x) is decreasing with V(x)=0 for $x \geq -c(0)m^{-1}$. It is Lipschitz continuous with $|V(x) - V(y)| \leq |x - y|$.

Proof:

It is clear that V(x) is decreasing. That V(x)=0 for $x \geq -c(0)m^{-1}$ follows as for the diffusion approximation. Let z> x , and let B={b $_t$ } be a reinsurance strategy for initial capital z such that $V^B(z) \leq V(z) + \varepsilon$. For initial capital x , choose the strategy $\tilde{B}$ (which is not optimal): inject the capital z- x and then follow the strategy B. Thus, $V(x) - V(z) \leq V^{\tilde{B}}(x) - V^B(z) + \varepsilon$ = $z - x + \varepsilon$ Because ε is arbitrary we have $|V(x) - V(z)| \leq |x - z|$, which proves the Lipschitz continuity. As a consequence, V(x) is absolutely continuous. We conjecture that the value function V(x) solves the HJB equation

$$\inf_{b \in [0,\tilde{b}]} \lambda \int_0^\infty V(x - r(z,b))dG(z) + (c(b) + mx)V'(x) - (\delta + \lambda)V(x) = 0 \cdot \qquad (6)$$

Remark:

(Optimal strategy at x =0) Suppose that the premium rate function c(b) is calculated by the expected value principle: $c(b) = \lambda(1+\theta)E[r(Z,b)] - \lambda\mu(\theta - \eta)$. Consider the initial capital x =0, and let b_0 be the root of the equation c(b)=0. Assume for the moment that a strategy b with $c(b) \leq 0$ is optimal at x =0. Since the surplus never leaves the value 0, we have $V(0) = E[\sum_{i=1}^{\infty} r(Z_i, b)e^{-\delta T_i}] - \frac{c(b)}{\delta} = \frac{\lambda E[r(Z,b)] - c(b)}{\delta}$

From the HJB equation (2.1) we obtain by rearranging the terms,

$$V(0) = \frac{\lambda}{\delta}[E[r(Z,b)] + \{(1+\theta)E[r(Z,b)] - (\theta - \eta)\mu\}V'(0)] \cdot$$

We conclude that V′(0)=-1. The implies that the right hand side of the above equation is decreasing in b; hence, $b = b_0$ would be optimal. In particular $V(0) = \frac{\lambda E[r(Z,b_0)]}{\delta}$. Let $k, \varepsilon > 0$ such that $k > c(\tilde{b})\varepsilon$. Consider the strategy $b_t = b_0 1_{\{t \geq T_1 \wedge \varepsilon\}}$. This strategy has the value bounded by

$$e^{-(\lambda+\delta)\varepsilon}\left(\frac{\lambda}{\delta}E[r(Z,b_0)] - \frac{\lambda}{\lambda+\delta}c(\tilde{b})\varepsilon(1 - G(k))\right) + \int_0^\varepsilon \left(\mu + \frac{\lambda}{\delta}E[r(Z,b_0)] - c(\tilde{b})t(1 - G(k))\right)\lambda e^{-(\lambda+\delta)t}dt$$

Taking the derivative with respect to ε shows that the function is decreasing in ε , with a derivative bounded away from 0. Thus, for small enough k and ε , the above strategy yields a smaller value than the strategy $b_t = b_0$. This shows that b_0 cannot be optimal. Equation (6) at x =0 reads

$$\inf_{b \in [0,\tilde{b}]} \lambda E[r(Z,b)][1 + (1+\theta)V'(0)] - \lambda\mu(\theta - \eta)V'(0) - \delta V(0) = 0$$

We see that the minimum is taken either at b=0 or $b = \tilde{b}$. Because b=0 is not optimal, we conclude that $b = \tilde{b}$. In particular, we obtain $V'(0) < \frac{-1}{(1+\theta)}$ and $V'(x) < \frac{-1}{(1+\theta)}$ for $x \in [0, y)$ and some y>0. By the continuity of the left-hand side of (6), we conclude that $b^*(x) = \hat{b}$ for small enough x . That is, no reinsurance is taken for capital close to 0.

Remark:

Consider the function $g_x(b) = (c(b) + mx)V'(x) - (\delta + \lambda)V(x) + \lambda\int_0^\infty V(x - r(z,b))dG(z)$. Let $b_1, b_2 \in [0, \tilde{b}]$ and $b_1 > b_2$. Then from the Lipschitz continuity of V(x) for every $x \in [0, \infty)$. We obtain

$$g_x(b_1) - g_x(b_2) = \lambda\int_0^\infty (V(x - r(z,b_1)) - V(x - r(z,b_2)))dG(z) + (c(b_1) - c(b_2))V'(x)$$

$$= \lambda\int_0^\infty (V(x - r(z,b_1)) - V(x - r(z,b_2)) + \{r(z,b_1) - r(z,b_2)\}(1+\theta)V'(x))dG(z)$$

$$\leq \lambda\int_0^\infty \{r(z,b_1) - r(z,b_2)\}[1 + (1+\theta)V'(x)]dG(z) \cdot$$

The condition $V'(x) \leq -\frac{1}{(1+\theta)}$ implies that $g_x(b)$ is decreasing, so that the minimum is taken in $b = \tilde{b}$, which is then the optimal strategy if V(x) is differentiable in x . On the other hand, if $b_0(x) \neq b < \tilde{b}$ is optimal for some $x \in [0, \infty)$ then it must hold that $V'(x) > \frac{-1}{(1+\theta)}$. We did not establish that the value function is continuously differentiable, even though the authors believe that this is actually the case. We therefore now give a sufficient condition for continuous differentiability.

Lemma:

If the value function V(x) is convex then V(x) is continuously differentiable.

Proof:

Let $b_0(x)$ denote the root of the equation c(b)+m x for $x \geq 0$, and define $f(x) := \lambda \int_0^\infty V(x - r(b_0(x), z))dG(z) - (\delta + \lambda)V(x)$. By (6) $f(x) \geq 0$. Let $V'(x-)$ and $V'(x+)$ denote the derivatives from the right and from the left, respectively. Assume now that there exists $\tilde{x} \in (0, \infty)$ with $V'(\tilde{x}-) < V'(\tilde{x}+)$. By Theorem $f(\tilde{x}) = 0$. Since in 0 we only consider the derivative from the right, we have $\tilde{x} > 0$. Without loss of generality, we can assume that V(x) is continuously differentiable on $(0, \tilde{x})$ which means that f(x) is continuously differentiable on $(0, \tilde{x})$. There exist sequences $(h_n)_{n \geq 0} \in (0, \tilde{x}), \lim_{n \to \infty} h_n = \tilde{x}$, with $f'(h_n) \leq 0$ and $(x_n)_{n \geq 0} \in (\tilde{x}, \infty), \lim_{n \to \infty} x_n = \tilde{x}$, with $f'(x_n) \geq 0$. Letting n tend to ∞ we obtain

$$\lambda \int_0^\infty V'(\tilde{x} - r(b_0(x), z))dG(z) - (\delta + \lambda)V'(\tilde{x}-) \leq 0,$$

$$\lambda \int_0^\infty V'(\tilde{x} - r(b_0(x), z))dG(z) - (\delta + \lambda)V'(\tilde{x}+) \geq 0.$$

Thus, $V'(\tilde{x}+) \leq V'(\tilde{x}-)$. which is a contradiction.

Examples:

Let us first note that in the case of proportional reinsurance we find, as in the case without an interest rate [2] that the value function is convex, provided that c(b) is concave. By Lemma 2.2 we can conclude that the value function is continuously differentiable. The problem in the numerical calculation of the value function is that we do not have the initial value V(0). But since we know V(x)=0 for $x \geq \lambda\mu(\theta - \eta)m^{-1}$, the calculation of the initial value becomes less complicated than in the case without an interest rate. Solve (2.1) with the initial value V_0. Let us denote the corresponding solution by $f(x; V_0)$ We then have to find V_0, such that $f(-m^{-1}c(0); V_0) = 0$. If $V_0 > V(0)$ then we would obtain $f(-m^{-1}c(0); V_0) > 0$. For a proof of the above statement, define $g(x) := f(x; V_0) - V(x)$ If $V_0 > V(0)$, we have g(0)>0. Let $b^*(x)$ denote the optimal strategy for V(x), and let $b_0(x)$ be the root of the equation c(b) + m x = 0. Replacing the optimal b for f(x) by $b^*(x)$ yields

$$(c(b^*(x)) + mx)g'(x) + \lambda \int_0^\infty g(x - b^*(x)z)dG(z) - (\delta + \lambda)g(x) \geq 0. \qquad (7)$$

Note that g(x)=g(0) for $x \leq 0$. Because g(0)>0 and $b^*(0) = 1$. it follows that g′(0)>0. Let $\hat{x} = \inf\{x : g'(x) \leq 0\}$. Because g(x) is increasing, we conclude that $b^*(x) > b_0(x)$ on $[0, \hat{x})$. From (6) and Lemma, we conclude that $b^*(\hat{x}) > b_0(\hat{x})$ also. Because g(x) is increasing on $[0, \hat{x}]$, it follows from (7) that $(c(b^*(\hat{x})) + m\hat{x})g'(\hat{x}) > 0$, which is a contradiction. So the function g(x) is strictly increasing on R_+ Therefore, $f(x; V_0)$ will ultimately be increasing.

Remark:

For the special case $\delta = 0$, we can show the existence of a strong (continuously differentiable) solution to the corresponding HJB equation.

$$\inf_{b \in [0, \tilde{b}]} \lambda \int_0^\infty V(x - r(z, b))dG(z) + (c(b) + mx)V'(x) - \lambda V(x) = 0$$

The starting point is to rewrite the HJB equation using Fubini's theorem i.e.

$$\lambda \int_{s(x,b)}^\infty r(z, b)dG(z) - \lambda (1 - G(s(x, b)))x = \lambda \int_x^\infty (1 - G(s(y, b)))dy,$$

Where $s(x, b) = \inf\{z : r(z, b) > x\}$

Numerical Examples:

Example 1:

Consider the parameters η=0.3, λ=μ=1, μ_2=2, δ=0.04, θ=0.8, and m=0.03. It is easy to verify that k=7.49, $\tilde{x}$ =0.441$\bar{6}$, $\lambda\mu(\theta - \eta)m^{-1} = 16.\bar{6}$ and that the optimal strategy on [0.441$\bar{6}$, 16.$\bar{6}$] is given by

$$b(x) = \frac{13.3 - 0.8x}{12.98}$$

x=0.4 $b(0.4) = \frac{13.3 - 0.8(0.4)}{12.98} = \frac{12.98}{12.98} = 1$

x =4.45 $b(4.45) = \frac{13.3 - 0.8(4.45)}{12.98} = \frac{9.74}{12.98}$

b(4.45) = 0.75

x=8.5 $b(8.5) = \frac{13.3 - 0.8(8.5)}{12.98} = \frac{6.5}{12.98}$

b(8.5) = 0.50

x=12.55 $b(12.55)=\frac{13.3-0.8(12.55)}{12.98}=\frac{3.26}{12.98}$

b(12.55) = 0.25

x=16.6 $b(16.6)=\frac{13.3-0.8(16.6)}{12.98}=\frac{0.02}{12.98}$

b(16.6) = 0

x	b(x)
0.4	1
4.45	0.75
8.5	0.50
12.55	0.25
16.6	0

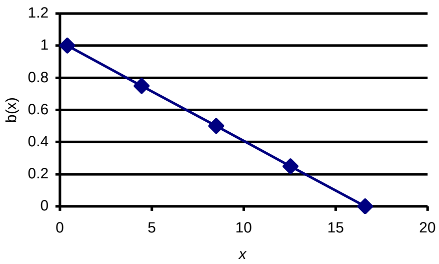

Figure 1: Optimal Strategy for b*(x)

Example 2:

(Proportional reinsurance for $Z_i \sim Exp(1/\mu)$). We consider here only the proportional reinsurance, i.e. r(z,b) = zb. All the considerations concerning the function $V^1(x)$ in the $\delta > 0$ case also hold in the $\delta = 0$ case. But here it is easier to consider the derivative $(V^1)'$. For the exponentially distributed claim sizes, $Z_i \sim Exp(1/\mu)$, we have to solve the integro-differential equation

$$-\lambda\int_0^x (V^1)'(y)e^{-(x-y)/\mu}dy + c(V^1)'(x) + \lambda\mu e^{-x/\mu} = 0\cdot$$

This equation is easy to solve, and we obtain as the derivative $(V^1)'(x) = -\frac{\lambda\mu}{c}\left(1+\frac{mx}{c}\right)^{\frac{\lambda}{m-1}} e^{-\frac{x}{\mu}}$ for $x \geq 0$.

Choose $\mu = \lambda = 1,\ m = 0.03,\ \theta = 0.5$ and $\eta = 0.3$. The optimal strategy for $Z_i \sim Exp(\frac{1}{\mu})$ is given.

C=-0.3 $(V^1)'(x) = -\frac{\lambda\mu}{c}\left(1+\frac{mx}{c}\right)^{\frac{\lambda}{m-1}} e^{-\frac{x}{\mu}}$

x=1 $(V^1)'(1) = \frac{-(1)(1)}{-0.3}\left(1+\frac{(0.03)(1)}{-0.3}\right)^{\frac{1}{0.03-1}} e^{-1/1} = 3.333(1-0.1)^{-1.030}e^{-1} = \frac{3.333}{(0.9)^{1.030}}(0.368)$

= (3.716)(0.368) = 1.37

x=2 $(V^1)'(2) = \frac{-(1)(1)}{-0.3}\left(1+\frac{(0.03)(2)}{-0.3}\right)^{\frac{1}{0.03-1}} e^{-2/1} = 3.333(1-0.2)^{-1.030}e^{-2} = \frac{3.333}{(0.8)^{1.030}}(0.135)$

= (4.192)(0.135) = 0.57

x=3 $(V^1)'(3) = \frac{-(1)(1)}{-0.3}\left(1+\frac{(0.03)(3)}{-0.3}\right)^{\frac{1}{0.03-1}} e^{-3/1} = 3.333(1-0.3)^{-1.030}e^{-3} = \frac{3.333}{(0.7)^{1.030}}(0.0497)$

= (4.8095)(0.0497) = 0.24

x=4 $(V^1)'(4) = \frac{-(1)(1)}{-0.3}\left(1+\frac{(0.03)(4)}{-0.3}\right)^{\frac{1}{0.03-1}} e^{-4/1} = 3.333(1-0.4)^{-1.030}e^{-4} = \frac{3.333}{(0.6)^{1.030}}(0.018)$

= (5.6395)(0.018) = 0.10

x=5 $(V^1)'(5) = \frac{-(1)(1)}{-0.3}\left(1+\frac{(0.03)(5)}{-0.3}\right)^{\frac{1}{0.03-1}} e^{-5/1} = 3.333(1-0.5)^{-1.030}e^{-5} = \frac{3.333}{(0.5)^{1.030}}(0.007)$

= (6.806)(0.007) = 0.05

x=6 $(V^1)'(6) = \frac{-(1)(1)}{-0.3}\left(1+\frac{(0.03)(6)}{-0.3}\right)^{\frac{1}{0.03-1}} e^{-6/1} = 3.333(1-0.6)^{-1.030}e^{-6} = \frac{3.333}{(0.4)^{1.030}}(0.002)$

=(8.568)(0.002) = 0.02

x=7 $(V^1)'(7) = \frac{-(1)(1)}{-0.3}\left(1+\frac{(0.03)(7)}{-0.3}\right)^{\frac{1}{0.03-1}} e^{-7/1} = 3.333(1-0.7)^{-1.030}e^{-7} = \frac{3.333}{(0.3)^{1.030}}(0.001)$

= (11.533)(0.001) = 0.01

x	$(v^1)'(x)$
1	1.37
2	0.57
3	0.24
4	0.10
5	0.05
6	0.02
7	0.01

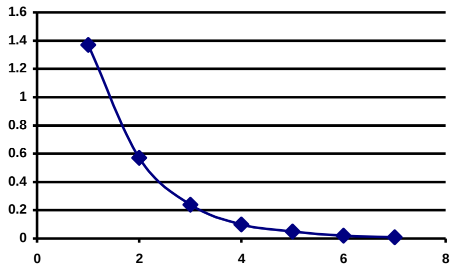

Figure 2: Optimal Strategy for $Z_i \sim Exp(1/\mu)$

C=-0.4 $(V^1)'(x) = -\frac{\lambda\mu}{c}\left(1+\frac{mx}{c}\right)^{\frac{\lambda}{m-1}} e^{-\frac{x}{\mu}}$

x=1 $(V^1)'(1) = \frac{-(1)(1)}{-0.4}\left(1+\frac{(0.03)(1)}{-0.4}\right)^{\frac{1}{0.03-1}} e^{-1/1} = 2.5(1-0.075)^{-1.030}e^{-1} = \frac{2.5}{(0.925)^{1.030}}(0.368)$

= (2.709)(0.368) = 0.997

x=2 $(V^1)'(2) = \frac{-(1)(1)}{-0.4}\left(1+\frac{(0.03)(2)}{-0.4}\right)^{\frac{1}{0.03-1}} e^{-2/1} = 2.5(1-0.15)^{-1.030}e^{-2} = \frac{2.5}{(0.85)^{1.030}}(0.135)$

= (2.955)(0.135) = 0.399

x=3 $(V^1)'(3) = \frac{-(1)(1)}{-0.4}\left(1+\frac{(0.03)(3)}{-0.4}\right)^{\frac{1}{0.03-1}} e^{-3/1} = 2.5(1-0.225)^{-1.030}e^{-3} = \frac{2.5}{(0.775)^{1.030}}(0.0497)$

= (3.251)(0.0497) = 0.16

x=4 $(V^1)'(4) = \frac{-(1)(1)}{-0.4}\left(1+\frac{(0.03)(4)}{-0.4}\right)^{\frac{1}{0.03-1}} e^{-4/1} = 2.5(1-0.3)^{-1.030}e^{-4} = \frac{2.5}{(0.7)^{1.030}}(0.018)$

= (3.608)(0.018) = 0.06

x=5 $(V^1)'(5) = \frac{-(1)(1)}{-0.4}\left(1+\frac{(0.03)(5)}{-0.4}\right)^{\frac{1}{0.03-1}} e^{-5/1} = 2.5(1-0.375)^{-1.030}e^{-5} = \frac{2.5}{(0.625)^{1.030}}(0.007)$

= (4.058)(0.007) = 0.03

x=6 $(V^1)'(6) = \frac{-(1)(1)}{-0.4}\left(1+\frac{(0.03)(6)}{-0.4}\right)^{\frac{1}{0.03-1}} e^{-6/1} = 2.5(1-0.45)^{-1.030}e^{-6} = \frac{2.5}{(0.55)^{1.030}}(0.002)$

= (4.6296)(0.002) = 0.01

x=7 $(V^1)'(7) = \frac{-(1)(1)}{-0.4}\left(1+\frac{(0.03)(7)}{-0.4}\right)^{\frac{1}{0.03-1}} e^{-7/1} = 2.5(1-0.525)^{-1.030}e^{-7} = \frac{2.5}{(0.475)^{1.030}}(0.001)$

= (5.376)(0.001) = 0.01

x	$(V^1)'(x)$
1	0.997
2	0.399
3	0.16
4	0.06

5	0.03
6	0.01
7	0.01

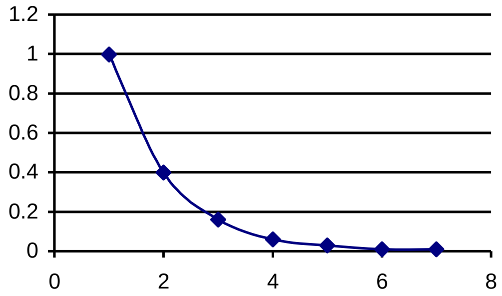

Figure 3: Optimal Strategy for $Z_i \sim Exp(1/\mu)$

Conclusion:

In our dissertation we have considered a classical risk model and its diffusion approximation, where the individual claims are reinsured by a reinsurance treaty with deductible. In addition the insurer is allowed to invest in a riskless asset with some constant interest rate $m > 0$. We have discussed, the concepts of (i) Proportional reinsurance for diffusion approximation, (ii) the classical risk model and have presented some definitions and examples. This project is used to minimise the expected discounted capital injections over all admissible reinsurance strategies and to find an explicit expression for the optimal strategy using Hamilton-Jacobi-Bellman equation.

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