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A STUDY ON FIXED POINT THEOREMS OF GENERALIZED CONTRACTIONS IN PARTIALLY ORDERED CONE METRIC SPACES

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Introduction:

Let E be a real Banach Space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies:

- ✓ P is closed, non-empty and $P \neq \{0\}$,
- ✓ $a, b \epsilon \mathbb{R}$, $a, b \geq 0$ and $x, y \epsilon P$ imply that $ax + by \epsilon P$,
- ✓ $x \epsilon P$ and $-x \epsilon P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; that is, $x \leq y$ if and only if $y - x \epsilon P$. Also we write $x \ll y$ if $y - x \epsilon P^0$, where P^0 denotes the interior of P. A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $||x|| \leq K||y||$. In the sequel, suppose that E is a real Banach space. P is a cone in E with nonempty interior $P^0 \neq 0$ and $\leq$ is the partial ordering with respect to P.

Definition: Let X be a nonempty set. Assume that the mapping $d: X \times X \to E$ satisfies

- ✓ $0 \leq d(x, y)$ for all $x, y \epsilon X$ and $d(x, y) = 0 \; iff \; x = y$,
- ✓ $d(x, y) = d(y, x)$ for all $x, y \epsilon X$,
- ✓ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \epsilon X$.

Then d is called a cone metric on X, and (X, d) is called cone metric space. The following remark will be useful in the sequel.

Remarks:

- ✓ If $u \leq v$ and $v \ll w$, then $u \ll w$.
- ✓ If $0 \leq u \ll c$ for each $c \epsilon P^0$, then $u = 0$.
- ✓ If $a \leq b + c$ for each $c \epsilon P^0$, then $a \leq b$.
- ✓ If $0 \leq x \leq y$, and $0 \leq a$, then $0 \leq ax \leq ay$.
- ✓ If $0 \leq x_n \leq y_n$ for each $n \epsilon \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ then $0 \leq x \leq y$.
- ✓ If $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then $d(x_n, x) \ll c$ where x_n, x are respectively, a sequence and a given point in X.
- ✓ If E is a real Banach Space with a cone P and if $a \leq \lambda a$ where $a \epsilon P$ and $0 < \lambda < 1$, then $a = 0$.
- ✓ If $c \epsilon P^0$, $0 \leq a_n$ and $a_n \to 0$, then there exists N such that for all $n > N$ and $a_n \ll c$.

Definition: An altering distance function is a function $\psi: P \to P$ which satisfies:

- ✓ ψ is continuous and non-decreasing.
- ✓ $\psi(x) = 0$ if and only if $x = 0$.

Definition: If $(X, \sqsubseteq)$ is a partially ordered set and $f: X \to X$, f is monotone non- decreasing if $x, y \epsilon X, x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$.

Definition: The cone P is called regular if every increasing sequence which is bounded form above is convergent. If $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some yϵE, then there is xϵE such that $\lim_{n\to\infty} ||x_n - x|| = 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been mentioned that every regular cone is normal.

Definition: P called minihedralcone if $sup\{x, y\}$ exists for all $x, y \epsilon E$, and strongly minihedral if every subset of E which is bounded above has a supremum. If cone P be strongly minihedral then, every subset of P has infimum.

Main Results: Let $(X, \sqsubseteq)$ be a partially ordered set and suppose there exists a cone metric $d \epsilon X$. Define (ID) property as follows, For all $x, y \epsilon X$ if there exists $z \epsilon X$ such that, $x \sqsubseteq y \sqsubseteq z$, then $d(x, y)$ and $d(y, z)$ are comparable.

Theorem:

Let $(X, \sqsubseteq)$ be a partially ordered set and suppose there exists a cone metric $d \epsilon X$ such that (X, d) is a complete cone metric space which the (ID) property holds. Let $f: X \to X$ be a continuous and non-decreasing mapping such that $\psi\big(d(fx, fy)\big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big)$ for $x \sqsubseteq y$, where ψ and φ are altering distance functions. If there exists $x_0 \epsilon X$ with $x_0 \sqsubseteq fx_0$ then f has a unique fixed point.

Proof:

If $x_0 = fx_0$ then the proof is finished. Suppose that $x_0 \neq fx_0$. Since $x_0 \sqsubseteq fx_0$ and f is a non-decreasing function, $x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq f^3x_0$ .... Put $x_{n+1} := fx_n = f^n x_0$ and $a_n := d(x_{n+1}, x_n)$. Then for $n \geq 1$, $\psi\big(d(x_{n+1}, x_n)\big) = \psi\big(d(fx_n, fx_{n-1})\big) \leq \psi\big(d(x_n, x_{n-1})\big) - \varphi(d(x_n, x_{n-1}))$, Therefore

$$0 \leq \psi(a_n) \leq \psi(a_{n-1}) - \varphi(a_{n-1}) \leq \psi(a_{n-1}). \quad (1)$$

Since $x_n \sqsubseteq x_{n+1} \sqsubseteq x_{n+2}$ by the (ID) property, It follows that

$$a_n \leq a_{n+1} \quad (2)$$

or

$$a_{n+1} \leq a_n. \quad (3)$$

If (2) holds, since ψ is non-decreasing by (1), It gives that

$$0 \leq \psi(a_n) \leq \psi(a_{n-1}) - \varphi(a_{n-1}) \leq \psi(a_n) - \varphi(a_{n-1}) \leq \psi(a_n) \quad (4)$$

This implies that $\varphi(a_{n-1}) = 0$ and so $a_{n-1} = 0$ for $n \geq 1$. Thus $x_n = x_{n-1} = fx_{n-1}$ for $n \geq 1$ are fixed points of f. If (3) holds, since ψ and φ are non-decreasing by, relation (1) and induction, It implies that

$$\begin{aligned}\varphi(a_{n+1}) \leq \varphi(a_n) \leq \psi(a_n) &\leq \psi(a_{n-1}) - \varphi(a_{n-1})\\ &\leq \psi(a_{n-1}) - \varphi(a_n)\\ &\leq \psi(a_{n-2}) - \varphi(a_{n-2}) - \varphi(a_n).\\ &\leq \psi(a_{n-2}) - 2\varphi(a_n) \leq \cdots\\ &\leq \psi(a_0) - n\varphi(a_n)\end{aligned}$$

Then $0 \leq \varphi(a_n) \leq \frac{1}{1+n}\psi(a_0)$ for all n. This implies that $\varphi(\lim_{n\to\infty} a_n)\ in\ P \cap -P$ and $\varphi(\lim_{n\to\infty} a_n) = 0$ and since φ is altering distance function. Then $(\lim_{n\to\infty} a_n) = 0$, and $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. (5)

Now show that the sequence $\{x_n\}$ is cauchy.

Claim:

For every $c\ in\ E$ and $c \gg 0$ there exists N such that $d(x_{n+2}, x_n) \ll c$ for every $n \geq N$. Choose $c \gg 0$, by (5) there exists N such that $d(x_{n+1}, x_n) \ll \frac{c}{2}$ for all $n \geq N$. It makes that $d(x_{n+2}, x_n) \leq d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) \ll c$ for every $n \geq N$. Therefore for some N and $d(x_{n+2}, x_n) \ll c$ for every $n \geq N$. Now by induction $d(x_{n+m}, x_n) \ll c$ for every $n \geq N$ and for all integer number $m \geq 1$. The sequence $\{x_n\}$ is Cauchy and since (X, d) is complete, and thus there exists $x^*\ in\ X$ such that $x_n \to x^*$ and on the other hand f is continuous and $x_{n+1} = fx_n$. Then $x^* = fx^*$. For uniqueness let $x = fx$ and $y = fy$, and $\psi\big(d(x,y)\big) = \psi\big(d(fx, fy)\big) \leq \psi\big(d(x,y)\big) - \varphi(d(x,y))$ The last inequality gives us $\varphi\big(d(x,y)\big) = 0$ and by property of the altering distance functions this implies $d(x,y) = 0$. Therefore $x = y$. In the next theorem, the (Id) property is replaced by strongly minihedrallity of the cone.

Theorem:

Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that thereexistsa cone metric $d\ in\ X$ with strongly minihedral cone P, such that $(X, d)$is a complete cone metric space. Let $f: X \to X$ be a continuous and non decreasing mapping such that $\psi\big(d(fx, fy)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big)$, for $x \sqsubseteq y$, where ψ and φ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$ then f has a fixed point.

Proof:

By the proof of the theorem (1.8) the sequence $\{\psi(a_n)\}$has infimum. Put $b = inf_n \psi(a_n)$. Then there exists $\{\psi\big(a_{n_k}\big)\}_k$such that $\psi(a_{n_k}) \to b$ as $k \to \infty$. Now by (1)

$$0 \leq \psi\big(a_{n_k}\big) \leq \psi\big(a_{n_k-1}\big) - \varphi\big(a_{n_k-1}\big) \leq \psi\big(a_{n_k-1}\big), \quad (6)$$

Letting $k \to \infty$, $b \leq b - \varphi(\lim_{k\to\infty} a_{n_k-1}) \leq b$, This implies that $\varphi\big(\lim_{k\to\infty} a_{n_k-1}\big) \in P \cap -P$ and $\varphi\big(\lim_{k\to\infty} a_{n_k-1}\big) = 0$. In the next corollary, the (ID) property is replaced and strongly minihedrality of the cone by regularity.

Corollary:

Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X with regular cone P such that (X, d) is a completecone metric space. Let $f : X \to X$ be a continuous and non decreasing mapping such that $\psi(d(fx, fy)) \leq \psi(d(x,y)) - \varphi(d(x,y))$, for $x \sqsubseteq y$, where ψ and φ are altering distance functions. If there exists $x_0 \in X$with $x_0 \sqsubseteq fx_0$, then f has a fixed point.

Proof:

By proofing of the theorem (1.8) and relation (1) the sequence $\{\psi(a_n)\}$is decreasing and bounded below and P is regular cone. Then $\varphi(\lim_{n\to\infty} a_n) = 0$ Now similar as the proof of the previous theorem the proof is completed. In the sequel, theorems (1.8),(1.9) and corollary (1.10) are still valid where $f$is not necessarily continuous, but the following hypothesis holds in X. "If $\{x_n\}$is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \sqsubseteq x$ for all $n \in N$".

Theorem:

Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric $d in\ X$ such that (X, d) is a complete cone metric space which the (ID) property holds. Let $f : X \to X$ be a non decreasing mapping such that $\psi(d(fx, fy)) \leq \psi(d(x,y)) - \varphi(d(x,y))$,for $x \sqsubseteq y$, where ψ and $\varphi$are altering distance functions.

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$ and X satisfies in following Condition if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \sqsubseteq x$ for all $n \in N$, then f has a fixed point.

Proof:

Following the proof of theorem (1.8) it is enough to prove that $fx^* = x^*$. Since $\{x_n\} \subset X$ is a non decreasing sequence and $x_n \to x^*$ as $n \to \infty$. Now by hypothesis $x_n \sqsubseteq x^*$ for all $n \in N$ and for all $c \gg 0$ there exists N such that $d(x_n, x) \ll c$ and $(d(x_{n+1}, fx^*)) = \psi(d(fx_n, fx^*)) \leq \psi\big(d(x_n, x^*)\big) - \varphi\big(d(x_n, x^*)\big) \leq \psi(c)$, for all $n \geq N$. Since ψ and φ are altering distance function if $n \to \infty$ and $0 \leq \psi(\lim_{n\to\infty} d(x_{n+1}, fx^*)) \leq \psi(c)$, for all $c \gg 0$. Thus $0 \leq \psi\left(\lim_{n\to\infty} d(x_n, fx^*)\right) \leq \psi\left(\frac{c}{m}\right)$, for all $c \gg 0$ and every $m \in N$. It gives that $\psi(\lim_{n\to\infty} d(x_{n+1}, fx^*)) = 0$ Then $\lim_{n\to\infty} d(x_{n+1}, fx^*) = 0$. Let $c \in E$ and $c \gg 0$ so there exists N such that $d(x_{n+1}, fx^*) \ll c$ for every $n \geq N$. Thus for some N, $d(x^*, fx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, fx^*) \ll c$, for every $n \geq N$. This implies that $0 \leq d(x^*, fx^*) \ll c$ for all $c \gg 0$. Then $d(x^*, fx^*) = 0$ and consequently $x^* = fx^*$. In what follows, a sufficient condition is given for the uniqueness of the fixed point in theorem and corollary. This condition is: "for $x, y \in X$ there exists $z \in X$ which is comparable to x and y". (7)

Theorem:

Adding condition (7) to the hypothesis of theorem (1.9) (resp. corollary (1.10)) we obtain uniqueness of the fixed point of f *is* obtained.

Proof:

Let $x, y \in X$ are fixed points. Two cases are distinguished.

Case 1: If x is comparable to y then $fx = x$ is comparable to $fy = y$ and $\psi(d(x,y)) = \psi(d(fx, fy)) \leq \psi(d(x,y)) - \varphi(d(x,y))$. The last inequality gives us $\varphi(d(x,y)) = 0$ and by altering distance functionsproperties this implies $d(x,y) = 0$ therefore $x = y$.

Case 2: If x is not comparable to y then there exists $z \in X$ comparable to x and y. Monotonicity of f implies that $f^n z$ is comparable to $f^n x = x$ and to $f^n y = y$, for $n = 0, 1, 2, \ldots$ Moreover,

$$\begin{aligned} \psi\big(d(x, f^n z)\big) &= \psi\big(d(f^n x, f^n z)\big) \\ &\leq \psi\big(d(f^{n-1}x, f^{n-1}z)\big) - \varphi\big(d(f^{n-1}x, f^{n-1}z)\big) \\ &= \psi(d(x, f^{n-1}z)) - \varphi(x, f^{n-1}z)) \\ &\leq \psi\,(d(x, f^{n-1}z)) \end{aligned} \tag{8}$$

according to regularity or strongly minihedrality of the cone P, there exists $b \in E$ such that $\psi(d(x, f^n z)) \to b$ as $n \to \infty$. Now altering distance functions properties ψ and φ

$$\psi\big(d(x, f^n z)\big) \leq \psi\big(d(x, f^{n-1}z)\big) - \varphi\big(d(x, f^{n-1}z)\big) \leq \psi(d(x, f^{n-1}z)),$$

Letting $n \to \infty$, $b \leq b - \varphi(\lim_{n\to\infty} d(x, f^{n-1}z)) \leq b$, This implies that $\varphi(\lim_{n\to\infty} d(x, f^{n-1}z)) \in P \cap -P$ Then $\varphi(\lim_{n\to\infty} d(x, f^{n-1}z)) = 0$. Thus $\lim_{n\to\infty} d(x, f^{n-1}z) = 0$. And similarly $d(y, f^n z) \to 0$. Let $c \gg 0$ and $c \in E$, there exists N such that $d(x, f^n z) \ll c$ and $d(y, f^n z) \ll c$ for all $n \geq N$. Now by triangle inequality $d(x,y) \leq d(x, f^n z) + d(f^n z, y) \ll 2c$, for all $n \geq N$. Namely $0 \leq d(x,y) \ll c$ for all $c \gg 0$. Then $d(x,y) = 0$ and $x = y$.

Ciric's Fixed Point Theorem in a Cone Metric Space:

Lemma:

Let (x, d) be a cone metric space, P be a normal cone. Let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $||d(x_n, x)|| \to 0$ as $n \to +\infty$.

Lemma:

Let (x, d) be a cone metric space, (x_n) be a sequence in X. If (x_n) is convergent, then it is a Cauchy sequence.

Lemma:

Let (x,d) be a cone metric space, P be a normal cone. Let (x_n) be a sequence in X. Then, (x_n) is a Cauchy sequence if and only if $\big||d(x_n, x_m)|\big| \to 0$ as $m, n \to +\infty$. $\mathcal{L}(E)$ is denoted the set of liner bounded operators on E, endowed with the following norm: $\| S \| = \sup_{x\in E, x\neq 0} \frac{\|Sx\|}{\|x\|}$, for all S in $\mathcal{L}(E)$. It is clear that if S in $\mathcal{L}(E)$, $\| Sx \| \leq \| S \| \| x \|$, for all x in E. By $I: E \to E$ is denoted the identity operator, $Ix = x$, for all x in X. If S in $\mathcal{L}(E)$, by S^{-1} in $\mathcal{L}(E)$ is denoted(if such operator exists) the operator defined by: $S^{-1}Sx = SS^{-1}x = x$, for all x in E.

Fixed Point Theorem)

Let (X, d) be a complete cone metric space, P be a normal cone with normal constant $k (k \geq 1)$. Suppose the mapping $T: X \to X$ satisfies the following constractive condition:

$$\begin{aligned} d(Tx, Ty) \leq A_1(x,y)d(x,y) + A_2(x,y)d(x,Tx) + A_3(x,y)d(y,Ty) \\ + A_4(x,y)d(x,Ty) + A_4(x,y)d(y,Tx), \end{aligned} \tag{9}$$

for all x, y in X, where $A_i: X \times X \to \mathcal{L}(E), i = 1,2,3,4$. Further, assume that for all $x, y\ in\ X$, $\exists\, \alpha \in [0,1/k) | \sum_{i=1}^{4} \| A_i(x,y) \| + \| A_4(x,y) \| \leq \alpha$ (10)

$$\exists\, \beta \in [0,1) \mid \| s(x,y) \| \leq \beta \tag{11}$$

$$(A_1(x,y)+A_2(x,y))(P) \subseteq P \tag{12}$$

$$A_2(x,y)(P) \subseteq P \tag{13}$$

$$A_4(x,y)(P) \subseteq P \tag{14}$$

$$\left(I - A_3(x,y) - A_4(x,y)\right)^{-1}(P) \subseteq P \tag{15}$$

Here, $S: X \times X \to \mathcal{L}(E)$ *is given by:* $S(x,y) = \left(I - A_3(x,y) - A_4(x,y)\right)^{-1}\left(A_1(x,y) + A_2(x,y) + A_4(x,y)\right)$,
For all x, y *in* X. Then, Thas a unique fixed point.

Proof:

Let $x \in X$ be arbitrary and define the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ by: $x_0 = x, x_1 = Tx_0, \ldots . x_n = Tx_{n-1} = T^n x_0, \ldots$ By (9) it gets that:

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq A_1(x_{n-1}, x_n)d(x_{n-1}, x_n) + A_2(x_{n-1}, x_n)d(x_{n-1}, x_n)$$
$$+A_3(x_{n-1}, x_n)d(x_n, x_{n+1}) + A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})$$
$$+A_4(x_{n-1}, x_n)d(x_n, x_n)$$
$$=\left(A_1(x_{n-1}, x_n) + A_2(x_{n-1}, x_n)\right)d(x_{n-1}, x_n)$$
$$+ A_3(x_{n-1}, x_n)d(x_n, x_{n+1}) + A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}).$$

Using the triangular inequality, it given that: $d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$, then

$$d(x_{n-1}, x_n) + d(x_n, x_{n+1}) - d(x_{n-1}, x_{n+1}) \text{ in } P.$$

From (14), it follows that $A_4(x_{n-1}, x_n)[\, d(x_{n-1}, x_n) + d(x_n, x_{n+1}) - d(x_{n-1}, x_{n+1})]$ *in* P,
And $A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) \leq A_4(x_{n-1}, x_n)d(x_{n-1}, x_n) + A_4(x_{n-1}, x_n)\, d(x_n, x_{n+1})$.
Then it implies that: $d(x_n, x_{n+1}) \leq (A_1(x_{n-1}, x_n) + A_2(x_{n-1}, x_n) + A_4(x_{n-1}, x_n))d(x_{n-1}, x_n) + (A_3(x_{n-1}, x_n)$

$$+A_4(x_{n-1}, x_n))d(x_n, x_{n+1}).$$

Then, $\left(I - A_3(x_{n-1}, x_n)A_4(x_{n-1}, x_n)\right)d(x_n, x_{n+1}) \leq (A_1(x_{n-1}, x_n) + A_2(x_{n-1}, x_n) + A_4(x_{n-1}, x_n))d(x_{n-1}, x_n)$.
Using (15), $d(x_n, x_{n+1}) \leq S(x_{n-1}, x_n)(d(x_{n-1}, x_n))$. (16)
It is not difficult to see that under hypotheses (12),(14) and (15), $S(x,y)(P) \subseteq P$, *for all* x, y *in* X.
Using this remark, (16) and proceeding by iterations,

$$d(x_n, x_{n+1}) \leq S(x_{n-1}, x_n)\, S(x_{n-2}, x_{n-1}) \ldots S(x_0, x_1)\, d(x_0, x_1),$$

Which implies by (11) that:

$$\|d(x_n, x_{n+1})\| \leq k \| S\left(x_{n-1}, x_n\right) \| \| S(x_{n-2}, x_{n-1}) \| \ldots \ldots \| S(x_0, x_1) \| \| d(x_0, x_1) \| \leq k\beta^n \| d(x_0, x_1) \|.$$

For any positive integer p, $d(x_n, x_{n+p}) \leq \sum_{i=1}^{p} d(x_{n+i-1}, x_{n+i})$, Which implies that:

$$\| d(x_n, x_{n+p}) \| \leq k \sum_{i=1}^{p} \| d(x_{n+i-1}, x_{n+i}) \|$$
$$\leq k^2 \sum_{i=1}^{p} \beta^{n+i-1} \| d(x_0, x_1 \|$$
$$\leq k^2 \frac{\beta^n}{1-\beta} \| d(x_0, x_1) \| \tag{17}$$

Since $\beta \in [0,1), \beta^n \to 0$ *as* $n \to +\infty$. So from (17) it follows that the sequence $(x_n)_{n \in N}$ is Cauchy. Since (X, d) is complete, there is a point $u \in X$ such that:

$$\lim_{n \to +\infty} d(Tx_n, u) = \lim_{n \to +\infty} d(x_n, u) = \lim_{n \to +\infty} d(Tx_n, x_{n+1}) = 0 \tag{18}$$

Now, using the contractive condition (9), $d(Tu, Tx_n) \leq A_1(u, x_n)\mathrm{d}(u, x_n) + A_2(u, x_n)\, d(u, Tu)$

$$+ A_3(u, x_n)d(x_n, x_{n+1}) + A_4(u, x_n)d(x_n, x_{n+1}) + A_4(u, x_n)d(x_n, Tu).$$

By the triangular inequality, $d(u, Tu) \leq d(u, x_{n+1}) + d(x_{n+1}, Tu)$

$$d(x_n, Tu) \leq d(x_n, Tx_n) + d(Tx_n, Tu).$$

By (13) and (14), $A_2(u, x_n)d(u, Tu) \leq A_2(u, x_n)(d(u, x_{n+1}) + d(x_{n+1}, Tu))$

$$A_4(u, x_n)d(x_n, Tu) \leq A_4(u, x_n)d(x_n, Tx_n) + A_4(u, x_n)d(Tx_n, Tu).$$

Then $d(Tu, Tx_n) \leq A_1(u, x_n) + d(u, x_n) + (A_2(u, x_n) + A_4(u, x_n))d(u, x_{n+1})$

$$+(A_2(u, x_n) + A_4(u, x_n))d(x_{n+1}, Tu) + (A_3(u, x_n) + A_4(u, x_n))d(x_n, x_{n+1})$$

Using (10), this inequality implies that: $\| d(Tu, Tx_n) \| \leq \frac{k\alpha}{1-k\alpha}(\| d(u, x_n) \| + \| d(u, x_{n+1}) \| + \| d(x_n, x_{n+1}) \|)$.
From (18), it follows immediately that: $\lim_{n \to +\infty} d(Tu, Tx_n) = 0$. (19)
Then, (18), (19) and the uniqueness of the limit imply that $u = Tu$, *then* u is a fixed point of T . and$T$ has least one fixed point $u \in X$. Now, if $v \in X$ is another fixed point of T , by (9),

$$d(u, v) = d(Tu, Tv) \leq A_1(u, v)d(u, v) + 2A_4(u, v)d(u, v),$$

Which implies that: $\| d(u, v) \| \leq k(\| A_1(u, v) \| + 2 \| A_4(u, v) \|) \| d(u, v) \| \leq k\alpha \| d(u, v) \|$,

$$(1 - k\alpha) \| d(u, v) \| \leq 0.$$

Since $0 \leq \alpha < \frac{1}{k}$,we get $d(u, v) = 0$, i.e., $u = v$. So the proof of the theorem is completed.

Corollary:

Let (X, d)be a complete metric space and $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ be a mapping satisfying the following contractive condition: $d(Tu, Tx_n) \leq a_1(x,y)d(x,y) + a_2(x,y)d(x, Tx) + (a_3(x,y)d(y, Ty) + a_4(x,y)\big(d(x, Ty) +$

$d(y,Tx)\big)$ *for all* $x,y \in X$, *where* $a_i: X \times X \to [0,+\infty), i = 1,2,3,4$ *and* $\sum_{i=1}^{4} \alpha_i(x,y) + \alpha_4(x,y) \leq \alpha$ *for each* $x,y \in X$ *and some* $\alpha \in [0,1)$.*Then,* T *has a unique fixedpoint.*

Proof:

Consider E = R (with the usual norm) and P = [0,+1). Then, (X, d) is a complete cone metric space and P is a normal cone with normal constant $k = 1$. For each $i = 1,2,3,4$, Define$A_i: X \times X \to \mathcal{L}(E)$ by:

$$A_i(x,y): t \in \mathbb{R} \to a_i(x,y)t,$$

for all $x,y \in X$. let us check now that all the required hypotheses of theorem (2.4) are satisfied.

$$d(Tx,Ty) \leq A_1(x,y)d(x,y) + A_2(x,y)d(x,T) + (A_3(x,y)d(y,Ty) + A_4(x,y)\big(d(x,Ty) + d(y,Tx)\big)$$

for all $x,y \in X$. Then, condition (2.1) of theorem (2.4) is satisfied. For all $i = 1, \dots, 4$,

$$\| A_i(x,y) \| = a_i(x,y), \text{for all } x,y \in X.$$

Then, $\sum_{i=1}^{4} \| A_i(x,y) \| + \| A_i(x,y) \| \leq \alpha$ *, for all* $x,y \in X$ and condition (10) of theorem (2.4) is satisfied. For all $x,y \in X, S(x,y)t = \frac{a_1(x,y)+a_2(x,y)+a_4(x,y)}{1-a_3(x,y)-a_4(x,y)} t$*, for all t in* $\mathbb{R}$. Then, for all $x,y \in X$,

$$\| S(x,y) \| = \frac{a_1(x,y) + a_2(x,y) + a_4(x,y)}{1 - a_3(x,y) - a_4(x,y)}$$

Since $\alpha \in [0,1) a_1(x,y) + a_2(x,y) + a_4(x,y) + \alpha a_3(x,y) + \alpha a_4(x,y) \leq \alpha, \quad \forall x,y \in X$. Then, $\| S(x,y) \| \leq \alpha, \forall x,y \in X$ and condition (2.3) of theorem (2.4) holds with $\beta = \alpha$.

For all $x,y \in X$, we have: $\big(I - A_3(x,y) - A_4(x,y)\big)^{-1} s = \frac{s}{1-a_3(x,y)-a_4(x,y)}$, $\forall\, s \in \mathbb{R}$.

Since $a_3(x,y) + a_4(x,y) < 1$ for all $x,y \in X$, then $s \geq 0 \Rightarrow \big(I - A_3(x,y) - A_4(x,y)\big)^{-1} s \geq 0$. Now, It is able to apply theorem (2.4) and then, T has a unique fixed point.

Metrizability of Cone Metric Spaces:

Theorem:

For every cone metric $D: X \times X \to E$ there exists metric $d: X \times X \to \mathbb{R}^+$ which is equivalent to D on X.

Proof:

Define $d(x,y) = \inf\{\| u \|: D(x,y) \leq u\}$. We shall to prove that d is an equivalent metric to D. If $d(x,y) = 0$ then there exists u_n such that$\| u_n \| \to 0$ and $D(x,y) \leq u_n$. And$u_n \to 0$ and consequently for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq N$. Thus for all $c \gg 0, 0 \leq D(x,y) \ll c$. Namely $x = y$. If $x = y$ then $D(x,y) = 0$ which implies that $d(x,y) \leq \| u \|$ for all $0 \leq u$. Put $u = 0$ it implies $d(x,y) \leq \| 0 \| = 0$, on the other hand $0 \leq d(x,y)$, Therefore $d(x,y) = 0$. It is clear that $d(x,y) = d(y,x)$. To prove triangle inequality, for $x,y,z \in X, \forall \epsilon > 0\ \exists u_1\ \| u_1 \| < d(x,z) + \epsilon, D(x,z) \leq u_1, \forall \epsilon > 0\ \exists u_2\ \| u_2 \| < d(z,y) + \epsilon, D(z,y) \leq u_2$. But $D(x,y) \leq D(x,z) + D(z,y) \leq u_1 + u_2$, Therefore $d(x,y) \leq \| u_1 + u_2 \| \leq \| u_1 \| + \| u_2 \| \leq d(x,z) + d(z,y) + 2\epsilon$. Since $\epsilon > 0$ was arbitrary so $d(x,y) \leq d(x,z) + d(z,y)$.

Claim: For all $\{x_n\} \subseteq X$ *and* $x \in X, x_n \to x$ in (X,d) ifand only if $x_n \to x$ in (X,D). $\forall n,m \in N\ \exists u_{nm}$ such that $\| u_{nm} \| < d(x_n,x) + \frac{1}{\text{m}}$, $D(x_n,x)u_{nm}$. *Put* $v_n := u_{nn}$ *then* $\| v_n \| < d(x_n,x) + \frac{1}{n}$ and $D(x_n,x) \leq v_n$. Now if $x_n \to x$ in (X,d) then $d(x_n,x) \to 0$ and $v_n \to 0$. Therefore for all $c \gg 0$ thereexists $N \in \mathbb{N}$ such that $v_n \ll c$ for all $n \geq \mathbb{N}$. This implies that $D(x_n,x) \ll c$for all $n \geq \mathbb{N}$. Namely $x_n \to x$ in(X,D). Conversely, For every real $\epsilon > 0$, choose $c \in E$ with $c \gg 0$ And $\| c \| < \epsilon$. Then there exists $N \in \mathbb{N}$ such that $D(x_n,x) \ll c$ for all $n \geq N$. This means that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n,x) \leq \|c\| < \varepsilon$ for all $n \geq N$. Therefore $d(x_n,x) \to 0$ as$n \to \infty$ so$x_n \to x$ *in* (X,d).

Example: Let $0 \neq a \in P \subseteq \mathbb{R}^n$ with $\| a \| = 1$ and for every $x,y \in \mathbb{R}^n$ define $D(x,y) = \begin{cases} a, & x \neq y; \\ 0, & x = y. \end{cases}$ Then D is a cone metric on $\mathbb{R}^n$and its equivalent metric d is $d(x,y) = \begin{cases} 1, & x \neq y; \\ 0, & x = y. \end{cases}$ Which is discrete metric.

Example: Let $a,b \geq 0$ and consider the cone metric $D: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ with $D(x,y) = (ad_1(x,y), bd_2(x,y))$ where $d_1, d_2$are metrics on$\mathbb{R}$. Then its equivalent metric is $d(x,y) = \sqrt{a^2 + b^2} \| \big(d_1(x,y), d_2(x,y)\big) \|$. In particular if$d_1(x,y) := |x - y|andd_2(x,y) := \alpha|x - y|$, where $\alpha \geq 0$. Then D is the same famous cone metric and its equivalent metric is $d(x,y) = \sqrt{1 + \alpha^2}|x - y|$.

Example: For $q > 0,\ b > 1, E = l^q, P = \{\{x_n\}_{n\geq 1} : x_n \geq 0$, for all $n\}$ and (X,ρ) a metric space, define $D: X \times X \to E$ which is the same cone metric by $D(x,y) = \left\{\left(\frac{\rho(x,y)}{b^n}\right)^{\frac{1}{q}}\right\}_{n\geq 1}$. Then its equivalent metric on X is

$$d(x,y) = \left\| \left\{\left(\frac{\rho(x,y)}{b^n}\right)^{\frac{1}{q}}\right\}_{n\geq 1} \right\|_{l^q} = \left(\sum_{n=1}^{\infty} \frac{\rho(x,y)}{b^n}\right)^{\frac{1}{q}} = \left(\frac{\rho(x,y)}{b-1}\right)^{\frac{1}{q}}.$$

Lemma:

Let $D, D^*: X \times X \to \mathbb{E}$ be cone metrics, $d, d^*: X \times X \to \mathbb{R}^+$ their equivalent metrics respectively and $T: X \to X$ a self map. If $D(Tx,Ty) \leq D^*(x,y)$, then $d(Tx,Ty) \leq d^*(x,y)$.

Proof:

By the definition of d^*, $\forall \epsilon > 0\ \exists v\ \| v \| < d^*(x,y) + \varepsilon$, $D^*(x,y) \leq v$. Therefore if $D(Tx,Ty) \leq D^*(x,y) \leq v$, then $d(Tx,Ty) \leq \| v \| \leq d^*(x,y) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary so $d(Tx,Ty) \leq d^*(x,y)$.

Example: Let $E := \mathbb{R}, P := \mathbb{R}^+$ and $D : X \times X \to E$ be a cone metric, $d: X \times X \to \mathbb{R}^+$its equivalent metric, $T: X \to X$ a self map and $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$defined by $\varphi(x) = \frac{x}{1+x}$. If $D^* := \varphi(D)$,then its equivalent metric is $d^* = \varphi(d)$,and if $D(Tx,Ty) \le \varphi(D(x,y)) = \frac{D(x,y)}{1+D(x,y)}$, then $d(Tx,Ty) \le \varphi(d(x,y)) = \frac{d(x,y)}{1+d(x,y)}$.

Definition: A self map φ on normed space X is bounded if $\| \varphi \| := \sup_{0 \neq x \epsilon X} \frac{\|\varphi(x)\|}{\|x\|} < \infty$.

Theorem:

Let $D: X \times X \to E$ be a cone metric, $d: X \times X \to \mathbb{R}^+$its equivalent metric, $T: X \to X$ a self map and $\varphi: P \to P$ a bounded map, then there exists $\psi: \mathbb{R}^+ \to \mathbb{R}^+$such that $D(Tx,Ty) \le \varphi\big(D(x,y)\big)$for every $x, y \in X$ implies $d(Tx,Ty) \le (\| D(x,y) \|)$ for all $x, y \in X$.Moreover if is decreasing map or φ is linear and increasing map then,$d(Tx,Ty) \le (d(x,y))$ for all $x, y \in X$.

Proof:

Put $\psi(t) := \sup_{0 \neq x \epsilon P} \left\| \varphi\left(\frac{t}{\|x\|} x\right) \right\|$for all $t \in \mathbb{R}^+$, and$\| \varphi(x) \| \ \psi(x)$ $for\ all\ x \in P$. Therefore if $D(Tx,Ty) \le \varphi\big(D(x,y)\big)$, then $d(Tx,Ty) \le \| \varphi\big(D(x,y)\big) \| \le \psi(\| D(x,y) \|)$. Now if ψ be decreasing map, bythe definition of d we have $d(x,y) \le \| D(x,y) \|$,and $d(Tx,Ty) \le \psi(\| D(x,y) \|) \le \psi\big(d(x,y)\big)$.If φ be a linear increasing map then $\psi(t) = t \| \varphi \|$. The definition of d implies $\forall \varepsilon > 0\ \exists v\ \| v \| < d(x,y) + \varepsilon$, $D(x,y) \le v$. Therefore if $D(Tx,Ty) \le \varphi(D(x,y)) \le \varphi(v)$, then we have $d(Tx,Ty) \le \| \varphi(v) \| \le \psi(\| v \|) \le \psi(d(x,y)) + \psi(\varepsilon)$. Since $\varepsilon > 0$ was arbitrary and $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0$, so $d(Tx,Ty) \le \psi(d(x,y))$. In the following summary of our results are listed.

Corollary:

Let D, D^* be cone metrics, d, d^* their equivalent metrics, $T: X \to X\ a\ map, \lambda \in [0, \frac{1}{2})\ and\ \alpha, \beta \in [0,1).\ For\ x, y \in X,$

- ✓ $D(Tx,Ty) \le \alpha D(x,y) \Rightarrow d(Tx,Ty) \le \alpha d(x,y)$.
- ✓ $D(Tx,Ty) \le \lambda\big(D(Tx,x) + D(Ty,y)\big) \Rightarrow d(Tx,Ty) \le \lambda(d(Tx,x) + d(Ty,y))$.
- ✓ $D(Tx,Ty) \le \lambda\big(D(Tx,y) + D(Ty,x)\big) \Rightarrow d(Tx,Ty) \le \lambda(d(Tx,y) + d(Ty,x))$.
- ✓ $D(Tx,Ty) \le \alpha\big(D(x,y) + \beta D(Tx,y)\big) \Rightarrow d(Tx,Ty) \le \alpha d(x,y) + \beta d(Tx,y)$.
- ✓ There exists$u \in \{D(x,y); D(x,Tx); D(y,Ty); \frac{1}{2}[D(x,Ty) + D(y,Tx)]$ Such that $D(Tx,Ty) \le \alpha u$ where $\alpha \in (0,1)$, then $d(Tx,Ty) \le \alpha \max\{d(x,y); d(x,Tx); d(y,Ty); \frac{1}{2}[d(x,Ty)\ d(y,Tx)]\}$.
- ✓ There exists $u \in \{D(x,y); D(x,Tx); D(y,Ty); \frac{1}{2} D(x,Ty); \frac{1}{2} D(y,Tx)$such that $D(Tx,Ty) \le \beta u$ where$\beta \in (0,1)$. Then$d(Tx,Ty) \le \beta \max\{d(x,y); d(x,Tx); d(y,Ty); \frac{1}{2}\ d(x,Ty)\ ; \frac{1}{2} d(y,Tx)\}$.
- ✓ There exists $u \in \{D(x,y); \frac{1}{2}[D(x,Tx) + D(y,Ty)]; \frac{1}{2}[D(x,Ty) + \frac{1}{2} D(y,Tx)\}$ such that $D(Tx,Ty) \le \beta u$ where $\beta \in (0,1)$, then $d(Tx,Ty) \le \beta \max\{d(x,y); \frac{1}{2}[d(x,Tx) + d(y,Ty)]; \frac{1}{2}[d(x,Ty)\ d(y,Tx)]\}$.
- ✓ If $D(Tx,Ty) \le a_1 D(x,y) + a_2 D(x,Tx) + a_3 D(y,Ty) + a_4 D(x,Ty) + a_5 D(y,Tx)$, Then $d(Tx,Ty) \le a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx)$
- ✓ If there exists $u \in \{D(x,y); D(x,Tx); D(y,Ty); D(x,Ty); D(y,Tx)\}$ such that $D(Tx,Ty) \le \frac{\beta}{2} u$, then $d(Tx,Ty) \le \frac{\beta}{2} \max\{d(x,y); d(x,Tx): d(y,Ty); d(x,Ty); d(y,Tx)\}$. Where $\beta \in (0,1)$.
- ✓ If $D(Tx,Ty) \le a_1 D(x,y) + a_2 D(x,Ty) + a_3 D(y,Ty) + a_4 [D(x,Ty) D(y,Tx)]$, Then $d(Tx,Ty) \le a_1 d(x,y) + a_2 d(x,Ty) + a_3 d(y,Ty) + a_4 [d(x,Ty) d(y,Tx)]$ where $a_1 + a_2 + a_3 + 2a_4 < 1$.
- ✓ There exist $m, n \in \mathbb{N}$ and $k \in [0,1)$ such that$D(T^m x, T^n y) \le kD(z,t)$, $\forall\ x, y \in X, z \neq t\ and\ z, t \in \{x, y, T^p x, T^q y\}$ where $1 \le p \le m\ and\ 1 \le q \le n$, then $d(T^m x, T^n y) \le kd(z,t)$.
- ✓ If $D(Tx,Ty) \le D^*(x,y)$, then$d(Tx,Ty) \le d^*(x,y)$.

Fixed Point Theorems in Partial Cone Metric Spaces:

A Cone metric space is Hausdorff and so has the property that any singleton is a closed subset of the space. In applications to computer science, especially to computer domains, a space induced by a distance function in which a singleton need not be closed is used. A partial cone metric space is such a space which might have a great application potential in computer science.

Definition: A partial cone metric on a nonempty set X is a function $p: X \times X \to E$ such that for all $x, y, z \in X$ (P1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$, (P2) $0 \le p(x,x) \le p(x,y)$, (P3) $p(x,y) = p(y,x)$, (P4)$p(x,y) \le p(x,z) + p(z,y) - p(z,z)$. A partial cone metric space is a pair (X,p) such that X is a nonempty set and p isa partial cone metric on X. It is clear that, if $p(x,y) = 0$, then from $(p1)$and $(p2)$ $x = y$. But if $x = y, p(x,y)$ may not be 0. A cone metric space is a partial cone metric space. But there are partial cone metric spaces which are not cone metric spaces. The following two examples illustrate such two partial cone metric spaces.

Example: $E = \mathbb{R}^2$, $P = \{(x,y) \in E: x, y \geq 0\}$, $X = \mathbb{R}^+$and $p: X \times X \to E$ defined by $p(x,y) = (max\{x,y\}, \alpha max\{x,y\})$ Where$\alpha \geq 0$is a constant. Then (X,p) is a partial cone metric space which is nota cone metric space.

Example: $E = l_1, P = \{\{x_n\} \in l_1 : x_n \geq 0\}$, Let $X = \{(x_n) : (x_n) \in (\mathbb{R}^+)^\omega \sum x_n < \infty\}$ where $(\mathbb{R}^+)^\omega$be the set of all infinitesequences over $\mathbb{R}^+$, and $p: X \times X \to E$ defined by $p(x,y) = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n, \ldots)$. where the symbol $\vee$ $denotes$ the maximum, $x \vee y = max\{x,y\}$. Then $(X,p)$is a partial cone metric space which is not a cone metric space. Let (X,d) be a partial cone metric space, $x \in X$, and A be a non-empty subsetof X. We modify the concepts of distance between the set A and the singleton $\{x\}$, and the distance between two subsets A and $B$of Xin the following: $p(x,A) = inf\{p(x,a): a \in A\}$, $p(A,B) = inf\{p(a,b): a \in A, b \in B\}$ Throughout this chapter (X,p) will denote a partial cone metric space.

Theorem:

Any partial cone metric space (X,p) is a topological space.

Proof:

For$c \in intP$ let $B_p(x,c) = \{y \in X : p(x,y) \ll c + p(x,x)\}$ and $\beta = \{B_p(x,c): x \in X, c \in intP\}$. Then $\tau_p = \{U \subset X: \forall x \in U, \exists B \in \beta x \in B \subset U\}$is a topology on X.

Theorem:

Let (X,p) be a partial cone metric space and P be a normal cone with normal constant K, then (X,p) is T_0.

Proof:

Suppose $p: X \times X \to E$is a partial cone metric, and suppose$x, y \in X$ with $x \neq y$, from (p1) and (p2) $p(x,x) < p(x,y)$ or $p(y,y) < p(x,y)$. Suppose$p(x,x) < p(x,y)$ and$0 < p(x,y) - p(x,x)$, $0 <\| p(x,y) - p(x,x) \| = \delta_x$. For$\delta_x > 0$, choose $c_x \in intP$with $\| c_x \| < \delta_x$. Then$x \in B_p(x, c_x)$ and $y \notin B_p(x, c_x)$. Consequently (X,p) partial cone metric space isT_0.

Definition: Let (X,p) be a partial cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in intP$ there is Nsuch that for all $n > N$, $p(x_n, x) \ll c + p(x,x)$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n). Denote by $\lim_{n\to\infty} x_n = x \; or \; x_n \to x \; (n \to \infty)$

Theorem:

Let (X,p) be a partial cone metric space,P be a normal cone with normal constant K. Let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $p(x_n, x) \to p(x,x)(n \to \infty)$.

Proof:

Suppose that (x_n) converges to x. For every real $\varepsilon > 0$, choose$c \in intP$ with$K \| c \| < \varepsilon$. Then there is N, for all$n > N$, $p(xn, x) \ll c + p(x,x)$. So that when $n > N$, $\| p(x_n, x) - p(x,x) \| \leq K \| c \| < \varepsilon$. This means $p(x_n, x) \to p(x,x)(n \to \infty)$. Conversely, suppose that $p(x_n, x) \to p(x,x)$ $(n \to \infty)$. For $c \in intP$, there is $\delta > 0$, such that $\| x \| < \delta$ implies $c - x \in intP$. For this δ there is N, such that for all $n > N$, $\| p(x_n, x) - p(x,x) \| < \delta$. and $c - [p(x_n, x) - p(x,x)] \in intP$. This means $p(x_n, x) - p(x,x) \ll c$. Therefore (x_n) converges to x. Note that let (X,p) be a partial cone metric space, P be a normal cone with normal constant K, if $p(x_n, x) \to p(x,x)$ $(n \to \infty)$ then $p(x_n, x) \to p(x,x)$ $(n \to \infty)$.

Lemma:

Let (x_n) be a sequence in partial cone metric space (X,p). If a point x is the limit of (x_n) and $p(y,y) = p(y,x)$then y is the limit point of (x_n).

Proof:

Suppose that $x_n \to x$ and $p(y,y) = p(y,x)$. Since for all $c \in intP$ there is N such that for all $n > N$ $p(x_n, y) \leq p(x_n, x) + p(x,y) - p(x,x) \ll c + p(y,y)atx_n \to y$.

Definition: Let (X,p) be a partial cone metric space. (x_n) be a sequence in X.(x_n) is Cauchy sequence if there is $a \in P$ such that for every $\varepsilon > 0$ there is N such that for all $n, m > N$ $\| p(x_n, x_m) - a \| < \varepsilon$.

Definition: A quasi-cone metric space on a nonempty X is a function $q: X \times X \to E$ such that for all $x, y, z \in X$; (i) $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$ (ii) $q(x,y) \leq q(x,z) + q(z,y)$ A quasi-cone metric space is a pair (X,q) such that X is a nonempty set and q is a quasi-cone metric on X. Each quasi-cone metric q on X generates a T_0 topology τq on X which has as a base the family of open q-balls $\{B_q(x,c): x \in X, c \in intP\}$, where $B_q(x,c) = \{y \in X: q(x,y) \ll c\} for \; all \; x \in X \; and \; c \in intP$.

Lemma:

If (X,p) be a partial cone metric space, then the function $dp : X \times X \to P$defined by $d_P(x,y) = p(x,y) - p(x,x)$is a quasi-cone metric space on X. If the quasi-cone metric topology τdp and partial cone metric topology τp, then $\tau p = \tau dp$.

Proof

Consider $x, y \in X$. Then $d_P(x,y) = p(x,y) - p(x,x) \in P$because of $p(x,x) \leq p(x,y)$. It is easy to see that d_P is a quasi-cone metric. Now show that $p = \tau dp$. Indeed, let $x \in X$ and $c \in intP$ and consider $y \in B_{d_p}(x,c)$. Then$d_P(x,y) = p(x,y) - p(x,x) \ll c$ and, $p(x,y) \ll c + p(x,x)$. Consequently $y \in B_P(x,c)$and $\tau dp \subseteq \tau p$. Conversely if $y \in B_p(x,c)$ implies that $p(x,y) \ll c + p(x,x)$. Thus $dp(x,y) \ll c, y \in$

$Bdp(x,c)$ and gives that$\tau p \subseteq \tau dp$. If P is a partial cone metric on X, then the $d: X \times X \to E$ given by $d(x,y) = d_P(x,y) + d_P(y,x)$is a cone metric on X.

Theorem:

Let (X,p) be a partial cone metric space. If (x_n) is a Cauchy sequence in (X,p), then it is a Cauchy sequence in the cone metric space (X,d).

Proof:

Let (x_n) be a Cauchy sequence in (X,p). There is $a \in P$ for every real $\varepsilon > 0$ there is N, for all $n,m > N$ $\| p(xn,xm) - a \| < \frac{\varepsilon}{4}$. $d(x_n,x_m) = 2(p(x_n,x_m) - a) - (p(x_n,x_n) - a) - (p(x_m,x_m) - a)$ There exist$n,m > N$, $\| d(x_n,x_m) \| < \varepsilon$. This means $d(x_n,x_m) \to 0 (n,m \to \infty)$. Finally we show that $\tau_p \subseteq \tau_d$. Indeed, let $x \in X$ and $c \in intP$ and consider $y \in B_d(x,c)$. Then $d(x,y) \ll c$ from $(p2)$ $p(x,y) - p(x,x) \leq d(x,y) \ll c$and, then $p(x,y) \ll c + p(x,x)$. Consequently $y \in B_p(x,c)$ and $\tau p \subseteq \tau d$.

Theorem:

Let (X,p) be a partial cone metric space, P be a strongly minihedral, $A \subset X$ and$a \in X$.Then$a \in \bar{A}$if and only if $p(a,A) = p(a,a)$**.**

Proof:

Suppose that $a \in \bar{A}$. Then for each $c \in int\,P$there is $x_c \in A$ such that $p(a,x_c) \ll c + p(a,a)$. Hence for each $c \in int\,P$ $\inf\{p(a,x): a \in A\} \leq p(a,a) + c$ Then $p(a,A) \leq p(a,a)$.For all $x \in A, p(a,a) \leq p(a,x)$, then $p(a,a) \leq p(a,A)$. Therefore $p(a,A) = p(a,a)$.Then for all $c \in int\,P$ there is $x_c \in A$ such that $p(a,xc) \ll p(a,a) + c$.Then $B(a,c) \cap A \neq \emptyset$ for all $c \in intP$. This implies $a \in \bar{A}$(note that not all partial cone metric spaces areT_1 space and also there are partial cone metric spaces which are not T_1 space). As an example, in the partial cone metric space in Example (4.2), the singleton point set $\{x\}$is not closed. Indeed, suppose that $y > x > 0$, Then $p(y,\{x\}) = p(y,x) = (max\{y,x\}, \alpha max\{y,x\}) = (y, \alpha y = p(y,y))$. Then $y \in \{x\}$. Since a topological space X is a T_1 space if and only if any singletonpoint set is closed, the partial cone metric space (X,p) is not T_1. A partial cone metric space complete if every Cauchy sequence is convergent.

Theorem:

Let (X,p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T: X \to X$ satisfies the contractive condition. $p(Tx,Ty) \leq cp(x,y)$ for all $x,y \in X$where$c \in (0,1)$ is a constant. Then T has a unique fixed point in X, and for any$x \in X$ iterative sequence $(T^n x)$ converges to the fixed point.

Proof:

Choose $x_0 \in X$. Define the sequence$(x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0 \dots, x_{n+1} = Tx_n = T^{n+1}x_0, \dots)$.
Then for $m > n$, $p(x_m,x_n) \leq p(x_m,,x_{m-1}) + p(x_{m-1},x_{m-2}) + \cdots + p(x_{n+2},x_{n+1})p(,x_{n+1},x_n) - \sum_{k=1}^{m-n-1} p(x_m - k, x_m - k)$

$$\leq (c^{m-1} + c^{m-2} \dots + c^n) p(x_1,x_0)$$
$$= c^n \frac{1-c^{m-n}}{1-c} p(x_1,x_0)$$
$$\leq c^n \frac{1}{1-c} p(x_1,x_0)$$
$$\| p(x_m,x_n) \| \leq c^n K \frac{1}{1-c} \| p(x_1,x_0) \|.$$

Thus $(T^n x)$ is a Cauchy sequence in(X,p) such that $\lim_{n,m\to\infty} p(T^n x_0, T^m x_0) = 0$. As (X,p) is complete there exists $x_0 \in X$ such that $(T^n x_0)$ converges to x and $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_n) = 0$. Now for any $n \in N$, then $p(T\,x,x) \leq p(Tx, T^{n+1}x_0) + p(T^{n+1}x_0, x) - p(T^{n+1}x_0, T^{n+1}x_0) \leq cp(x,T^n x_0) + p(T^{n+1}x_0,x)$

$$\| p(T\,x,x) \| \leq Kc \| p(x,T^n x_0) \| + \| p(T^{n+1}x_0,x) \| \to 0$$

Hence $p(Tx,x) = 0$. But since $p(Tx,Tx) \leq cp(x,x) = 0$
$p(Tx,Tx) = p(Tx,x) = p(x,x) = 0$ which implies that $Tx = x$. Now if y is another fixed point of T, then $p(x,y) = p(Tx,Ty) \leq cp(x,y)$ Since $c < 1, p(x,y) = p(x,x) = p(y,y)$. Then $x = y$, thus the fixed pointof T is unique.

Theorem:

Let (X,p) be a complete cone partial metric space, P a normal conewith normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition $p(Tx,Ty) \leq k(p(Tx,x) + p(Ty,y))$ for all $x,y \in X$ where $k \in (0,12\ )$ is a constant. Then T has a unique fixed point X. And for any $x \in X$, iterative sequence $(T^n x)$ converges the fixed point.

Proof:

Choose $x_0 \in X$. Define the sequence $(x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \dots, x_{n+1} = Tx_n = T^{n+1}x_0, \dots)$

$$p(xn+1,xn) = p(T\,x_n, T\,x_{n-1}) \leq k\big(p(T\,x_n,x_n) + p(Tx_{n-1},x_{n-1})\big)$$
$$= k\big(p(x_{n+1},x_n) + p(x_n,x_{n-1})\big) \quad p(x_{n+1},x_n) \leq \frac{k}{1-k} p(x_n,x_{n-1}) = cp(x_n,x_{n-1})$$

Where c= $\frac{k}{1-k}$. For $m > n$;

$$p(x_m, x_n) \leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \cdots + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - \sum_{k=1}^{m-n-1} p(x_{m-k}, x_{m-k})$$
$$\leq p(x_m, x_{m-1}) + p((x_{m-1}, x_{m-2}) + \ldots + p(x_{n+2}, x_{n+1})\, p(x_{n+1}, x_n)$$
$$\leq [(c^{m-1} + c^{m-2} + \ldots + c^n)]p(x_1, x_0) \leq \frac{c^n}{1-c} p(x_1, x_0)$$
$$\| p(x_n, x_m) \| \leq \frac{c^n}{1-c} K \| p(x_1, x_0) \|.$$

This implies $p(x_n, x_m) \to 0\ (n, m \to \infty)$. There exist a Cauchy sequence (x_n). By the completeness of X, there is $x \in X$ suchthat $x_n \to x(n \to \infty)$, $\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n) = 0$ Since $p(Tx, x) \leq p(Tx, Tx_n) + p(Tx_n, x) - p(Tx_n, Tx_n) \leq k[p(Tx, x) + p(Tx_n, x_n)] + p(x_{n+1}, x)p(Tx, x)$

$$\leq \frac{1}{1-k}[kp(Tx_n, x_n) + p(x_{n+1}, x)]$$
$$p(Tx, x)k \leq K \frac{1}{1-k}(k \| p(x_{n+1}, x_n) \| + \| p(x_{n+1}, x) \|) \to 0$$

Hence $p(Tx, x) = 0$. But since $p(Tx, Tx) \leq k[p(Tx, x) + p(Tx, x)] = 2kp(Tx, x) = 0$ $p(Tx, Tx) = p(Tx, x) = p(x, x) = 0$ which implies that $Tx = x$. So xis fixed point of T. Now if y is another fixed point of T , Then $p(x, y) = p(Tx, Ty) \leq k[p(Tx, x) + p(Ty, y)] = 0$

$$p(x, y) = p(x, x) = p(y, y) = 0.$$

Therefore, $x = y$. The fixed point of T isunique.

Conclusion:

The present work contains not only an improvement and a generalization of the concept of a partial metric, as it has been presented in a more general setting, a partial cone metric space which is more general than the partial metric space, but also an investigation of some fixed point theorems one of which is also new for a partial metric space. So that one may expect it to be more useful tool in the field of topology in modeling various problems occurring in many areas of science, computer science, information theory, and biological science. On the other hand, a concept of fuzzy partial cone metric is investigated fixed points theorems for fuzzy functions. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present results.

References:

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