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A STUDY ON UNICYCLIC GRAPH

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Abstract:

The harmonic index of a graph G is defined as the sum of the weight. In this paper we present the minimum harmonic indices for unicyclic graphs.

Key Words: Harmonic Index, Diameter &Unicyclic Graph

Introduction:

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex v of G is denoted by $d(v)$. If $u, v \in V(G)$, then the distance between u and vis the length of a shortest $u – v$ path in G. The eccentricity of a vertex v is the greatest distance from v to any other vertex of G. The diameter of a graph is the maximum over eccentricities of all vertices of the graph and it is denoted by $D(G)$. For a graph G, the harmonic index $H(G)$ is defined as $H(G) = 2/(d(u)+d(v))$ we know, this index first appeared in [4]. Zhong found the minimum and maximum values of the harmonic index for simple connected graphs, trees and unicyclic graphs and characterized the corresponding external graphs [8][9]. Wu et al.gave a best possible lower bound for the harmonic index of a triangle- free graph with minimum degree at least two and characterized the extremal graphs [7]. Den et al. considered the relation connecting the harmonic index $H(G)$ and the chromatic number $\chi(G)$and proved that $\chi(G) \leq 2\, H(G)$ by using the effect of removal of a minimum degree vertex on the harmonic index[3]. Mehdi Sabzevariet al. gave the exact formula for Merrifield Simmons and Hosoya indices of some special graphs namely ladder graph, prism graph and book graph [6]. Zohreh Bagheriaet al. computed the edge-Szeged and vertex- PI indices of some important classes of benzenoid systems [10]. Liuproved that $H(T) - D(T) \geq 5/6 - n/2and(H(T))/(D(T)) \geq 1/2 + 1/(3(n-1))$for n-vertex tree T with equality for path and proposed it as a conjecture for any connected graph of order n[5]. The first part of the above conjecture was proved in [1] for unicyclic graphs. In this work, we prove the second part of the conjecture viz. $(H(G))/(D(G)) \geq 1/2 + 4/(3(n-4))$ for $n \geq 7$, when G is unicyclic graph. We conclude this section with some notations and terminology. Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. If $d(v) = 1$, then v is said to be a pendant vertex of G. The edge incident with v is referred to as pendant edge and the vertex adjacent to v is referred as the support vertex of v. The set of neighbors of v is denoted by $N(v)$. A diametrical path of a graph is a shortest path whose length this equal to the diameter of the graph. As usual, Cn and Pn denote the path on n vertices, respectively. In a cycle Cn, two vertices, say u and v are said to be Diametricallyopposite,ifd$(u, v) = n/2$when n is even and $d(u, v) = (n-1)/2$ when n is odd. Let $(U_n)^{(x,y)}$ be a unicyclic graph obtained from a cycle Cl by attaching two paths Px and Py to two diametrically opposite vertices of Cl such that $n = l+x+y$ For other notations in graph theory, maybe consulted[2].

Basic Results:

Lemma 1:

The function $f(x) = 1/(u+x) - 1/(u+x-1)$ is an increasing function on x for $x \geq 1$ and $u \geq 0$.

Lemma 2:

Let v be a pendent vertex of a connected graph G. then $H(G) > H(G-v)$.

Proof:

Let u be the support vertex of v. Then

$$H(G) - H(G-v) = 2/(d(u)+1) + 2\sum_{w \in N(u)-\{v\}} 1/(d(u)+d(w)) - 1/(d(u)+d(w)-1)$$
$$\geq 2/(d(u)+1) + 2\,(d(u)-1)\, \Sigma\, 1/(d(u)+1) - 1/d(u) \text{ by lemma 1}$$
$$= 2/(d(u)(d(u)+1))$$
$$>0$$

Hence $H(G) > H(G-v)$.

Analyzing the unicyclic graphs and its diametrical path, we have the following observation.

Observation:

If $G \cong Cn$ is a unicyclic graph on n vertices, then at least one of the end vertices of the diametrical path of G must be a pendant vertex.

Main Result:

In this section, we give the sharp lower bound of the relationship involving the harmonic index and diameter of connected unicyclic graphs.

Theorem 1:

Let G be a unicyclic graph of order $n \geq 7$ and diameter $D(G)$. Then $(H(G))/(D(G)) \geq 1/2 + 4/(3(n-4))$, where equality holds if and only if $G \cong (U_{n,4})^{1,n-5}$.

Proof:

Case 1: Let $G \cong C_n$ then $H(G) = n/2$, if n is even, then $D(G) = n/2$.
Hence $\frac{H(G)}{D(G)} = 1 \geq \frac{1}{2} + \frac{4}{3(n-4)}$. If n is odd, then $D(G) = \frac{n-1}{2}$. Hence
$\frac{H(G)}{D(G)} = 1 + \frac{1}{n-1} \geq \frac{1}{2} + \frac{4}{3(n-4)}$.

Case 2: Let $G \neq C_n$. Then G has at least one pendant vertex. Also by the observation, at least one of the end vertices of the diametrical path of G is a pendant vertex. Let P be a diametrical path of G. Now continue to remove pendant vertices from G so that P remains its diametrical path. Let the resulting graph be Gand v1, v2, . . . ,vk be the vertices in the order they were deleted. Thenwehave,

$$H(G) > H(G-v_1) > \cdots > H(G - \sum_{i=1}^{n} v_i) = H(G')$$

By Lemma 2 and

$$D(G) = D(G-v_1) = \cdots = D(G - \sum_{i=1}^{n} v_i) = D(G').$$

Clearly G' is also a unicyclic graph consisting of a cycle of length l together with at most two pendant paths, say P_x and P_y incident with two vertices of C_l, say u and v, such that $n = k+l+x+y$.

Sub Case 2.1: Let x = 0 and y = 1. In this case, $G' \cong (U_{n-k,\, n-k-l})^{0,1}$

$H(G') = \frac{n-k}{2} - \frac{1}{5}$. If l is even, then $D(G') = \frac{n-k+1}{2}$. Hence

$$\frac{H(G')}{D(G')} = \frac{5n-5k-2}{5(n-k+1)} = 1 - \frac{7}{5(n-k+1)} \geq \frac{1}{2} + \frac{4}{3(n-4)} \quad \text{since } n-k \geq 5.$$

If lis odd, then $D(G') = \frac{n-k}{2}$. Hence

$$\frac{H(G')}{D(G')} = \frac{5n-5k-2}{5(n-k)} = 1 - \frac{2}{5(n-k)} \geq \frac{1}{2} + \frac{4}{3(n-4)} \quad \text{since } n-k \geq 4.$$

Sub Case 2.2: Let x = 0 and $y \geq 2$. In this case, $H(G') = \frac{n-k}{2} - \frac{2}{15}$. If l is even, then $D(G') = \frac{n-k+y}{2}$. Hence

$$\frac{H(G')}{D(G')} = \frac{15n-15k-4}{15(n-k+y)} = 1 - \frac{15y+4}{15(n-k+y)} = \frac{1}{2} + \frac{15l+8}{30(2(n-k)-l)} \geq \frac{1}{2} + \frac{4}{3(n-4)} \quad \text{since } n-k = l+y \text{ and } l \geq 4.$$

If l is odd, then $D(G') = \frac{n-k+y-1}{2}$. Hence

$$\frac{H(G')}{D(G')} = \frac{15n-15k-4}{15(n-k+y-1)} = 1 - \frac{15y-11}{15(n-k+y-1)} = \frac{1}{2} + \frac{15l+7}{30(2(n-k)-l-1)} \geq \frac{1}{2} + \frac{4}{3(n-4)} \quad \text{since } n-k = l+y \text{ and } l \geq 3$$

Sub Case 2.3: Let x = 1, y =1.If u and v are nonadjacent, then $G' \cong (U_{n-k,\, l})^{1,1}$
Clearly $H(G') = \frac{n-k}{2} - \frac{2}{5}$. If lis even, then $D(G') = \frac{n-k}{2} + 1$. Hence

$$\frac{H(G')}{D(G')} = \frac{5n-5k-4}{5(n-k+2)} = 1 - \frac{14}{5(n-k+2)} = 1 - \frac{14}{5(l+4)} \quad \text{since } n-k = l+2$$
$$\geq \frac{1}{2} + \frac{4}{3(n-4)}.$$

If l is odd, then $D(G') = \frac{n-k+1}{2}$ .Hence

$$\frac{H(G')}{D(G')} = \frac{5n-5k-4}{5(n-k+1)} = 1 - \frac{9}{5(n-k+1)} = 1 - \frac{9}{5(l+3)} \quad \text{since } n-k = l+2$$
$$\geq \frac{1}{2} + \frac{4}{3(n-4)}.$$

Sub Case 2.4: Let x= 1 and $y \geq 2$. If u and v are adjacent.
Clearly $H(G') = \frac{n-k}{2} - \frac{3}{10}$and $D(G') = y + 2$. Hence

$$\frac{H(G')}{D(G')} = \frac{5n-5k-3}{10(y+2)} \geq \frac{5n-5k-3}{10(n-2)} = 1 - \frac{5n+5k-17}{10(n-2)} \geq \frac{1}{2} + \frac{4}{3(n-4)}.$$

If u and v are nonadjacent, then $G' \cong U_{n-k,l}^{l,y}$.Clearly $H(G') = \frac{n-k}{2} + \frac{1}{3}$. If l is even, then $D(G') = \frac{n-k+y+1}{2}$. Hence

$$\frac{H(G')}{D(G')} = \frac{3n-3k-2}{3(n-k+y+1)} = 1 - \frac{3y+5}{3(n-k+y+1)} \geq \frac{1}{2} + \frac{4}{3(n-4)}.$$

If l is odd, then $D(G') = \frac{n-k+y}{2}$. Hence

$$\frac{H(G')}{D(G')} = \frac{3n-3k-2}{3(n-k+y)} = 1 - \frac{3y+2}{3(n-k+y)} \geq \frac{1}{2} + \frac{4}{3(n-4)}.$$

Sub Case 2.5: Let x≥ 2 and y≥ 2. If u and v are adjacent, then H(G') = (n-k)/2 + 7/30 and D(G') = x+ y+ 1 = n− k− l+ 1. Hence

(H(G'))/(D(G^')) = (15n-15k-7)/(30(n-k-l+1))
≥ 1/2 + 4/(3(n-4))

If u and v are nonadjacent, then H(G') = H((U_(n-k,l))^(X,y)) =(n-k)/2 + 4/15 and D(G') ≤ D((U_(n-k,l))^(X,y)). If l is even D(G') ≤(n-k+x+y)/2. Hence

(H(G'))/(D(G^')) = (15n-15k-8)/(15(n-k+x+y))
= (15n-15k-8)/(15(2(n-k)-l))
= 1/2 +(15l-16)/(30(2(n-k)-l))
≥ 1/2 + 4/(3(n-4)) .

If lis odd. D(G') ≤(n-k+x+y-1)/2.Hence

(H(G'))/(D(G^'))>(15n-15k-8)/(15(n-k+x+y-1))
= (15n-15k-8)/(15(2(n-k)-l-1))
= 1/2 + (15l-1)/(30(2(n-k)-l-1))
≥ 1/2 + 4/(3(n-4)) .

For proving the equality, assume that (H(G))/(D(G))=1/2+4/(3(n-4)). Since D(G) ≤ n-2, (H(G))/(n-2)≤(H(G))/(D(G))for all t.
So our search is to find that tt, forwhichD(G) = n-2 and (H(G))/(D(G)) = 1/2 + 4/(3(n-4)).s(U_(n,3))^(0,n-3), (U_(n,3))^(1,n-4) , (U_(n,3))^(2,n-5),
(U_(n,4))^(0,n-4), (U_(n,4))^(1,n-5) and (U_(n,4))^(2,n-6)are the unicyclic graphs with D(tt) = n− 2. But (U_(n,4))^(1,n-5) is the only graphthat satisfies the equality. Hence G∼=(U_(n,4))^(1,n-5)and it is easy to check(H((U_(n,4))^(1,n-5)))/(D((U_(n,4))^(1,n-5))) = 1/2+4/(3(n-4))
Remark3.1.If n≤ 6, this lower bound is not true. One such graph,

(H(G))/(D(G)) =13/20≤2/3= 1/2 + 4/(3(n-4)).

This result seems true for any connected graph of order n, that is not a tree, and we propose it as a conjecture as follows.
Conjecture 1.Let G be a simple connected graph, that is not a tree, of order
n≥ 7 and diameter D(G). Then H(G) - D(G) ≥5/2 - n/2 and (H(G))/(D(G))≥1/2 + 4/(3(n-4)),
Where equality holds if and only if tt∼=(U_(n,4))^(1,n-5).

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